Le système hétérotique G_2 sur des 7–variétés de contact Calabi-Yau

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A brief word from our sponsors

$(n=4)$ Classical ASD instanton equation

$$*(F_A) = \pm F_A \tag{1}$$

Remark: instantons are YM-connections, i.e., $d_A(*F_A) = 0$

$(\ n>4)$ Higher dimensional instantons

(1) makes no sense, choose $\sigma \in \Upsilon^{n-4}(M)$, A is σ-**instanton** if

$$L_\sigma(F_A) := *(\sigma \wedge F_A) = \lambda F_A, \quad \lambda \in \mathbb{R} \tag{2}$$

YM-equation with torsion: $\lambda d_A(*F_A) = d\sigma \wedge F_A$
Remark: In general $d\sigma \neq 0$.

Examples

- Classical Donaldson theory: $\sigma = \pm 1$ on a oriented 4-manifold
- Hermitian-Einstein connections: $\sigma = \frac{1}{(m-2)!}\omega^{m-2}$ on a Kähler manifold of dimension $n = 2m$
- G_2-instantons: $\sigma = \varphi$ the 3-form preserved by G_2.
- $Spin(7)$-instantons: $\sigma = \Phi$ the 4-form preserved by $Spin(7)$

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Torsion of a G_2-structure

(K^7, φ) a G_2-structure manifold, $\psi = *\varphi \in \Omega^4(K)$, irreducible G_2-reps:

$$\Omega^2_{14}(K) = \{\beta \in \Omega^2(K) \,:\, \beta \wedge \varphi = - * \beta\} = \{\beta \in \Omega^2(K) \,:\, \beta \wedge \psi = 0\},$$
$$\Omega^3_{27}(K) = \{\gamma \in \Omega^3(K) \,:\, \gamma \wedge \varphi = 0, \, \gamma \wedge \psi = 0\}.$$

Definition (Torsion forms)

The *torsion* of φ is completely described by:
$\tau_0 \in C^\infty(K)$, $\tau_1 \in \Omega^1(K)$, $\tau_2 \in \Omega^2_{14}(K)$ and $\tau_3 \in \Omega^3_{27}(K)$, such that

$$d\varphi = \tau_0 \psi + 3\tau_1 \wedge \varphi + *\tau_3 \quad \text{and} \quad d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi.$$

Heterotic G_2 system: degrees of freedom

Given a smooth G-bundle $F \to K$, for some compact semi-simple Lie group G, let $\mathcal{A}(F)$ denote its space of smooth G-connections.

Geometric and gauge fields

The *heterotic* G_2 *system* or G_2*-Hull–Strominger system* on a 7-manifold with G_2-structure (K, φ) is comprised of the following degrees of freedom:

- Geometric fields:

$$\lambda \in \mathbb{R} \text{ (scalar field)}, \quad \mu \in C^\infty(K) \text{ (dilaton)}, \quad \text{and} \quad H \in \Omega^3(K) \text{ (flux)}.$$

- Gauge fields:

$$A \in \mathcal{A}(E), \quad \text{and} \quad \theta \in \mathcal{A}(TK),$$

where $E \to K$ is a vector bundle and both connections are respectively G_2-instantons (possibly up to $O(\alpha')$ related to string scale):

$$F_A \wedge \psi = 0 \quad \text{and} \quad R_\theta \wedge \psi = O(\alpha')^2.$$

Heterotic G_2 system: constraints

Heterotic G_2 problem [de la Ossa, Larfors, Svanes 2015-2018]

The geometric fields satisfy the following relations with the torsion of the G_2-structure φ:

$$\tau_0 = \frac{3}{7}\lambda \qquad \tau_1 = \frac{1}{2}d\mu \qquad \tau_2 = 0$$

$$H =: \frac{\lambda}{14}\varphi \oplus H^{\perp} = \frac{1}{6}\tau_0\varphi - \frac{1}{2}d\mu^{\#}\lrcorner\psi - \tau_3 \qquad \tau_3 = -H^{\perp} - \frac{1}{2}d\mu^{\#}\lrcorner\psi. \tag{3}$$

The flux compensates exactly the Chern–Simons defect between the gauge fields via the *heterotic Bianchi identity*:

$$\mathrm{d}H = \frac{\alpha'}{4}\left(\mathrm{tr}\, F_A \wedge F_A - \mathrm{tr}\, R_\theta \wedge R_\theta\right), \tag{4}$$

where F_A is the curvature of A, R_θ is the Riemann curvature of θ.

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Sasakian geometry

Contact manifold (M^{2n+1}, η, ξ)

- A **contact form** $\eta \in \Omega^1(M)$ s.t. $\eta \wedge d\eta^n \neq 0$.
- $\xi \in \mathfrak{X}(M)$ s.t. $i_\xi \eta = 1$, $i_\xi d\eta = 0$.

NB.: The contact structure splits $TM = \ker(\eta) \oplus \mathbb{R} \cdot \xi = H \oplus N_\xi$.
Contact instantons: $\sigma = \eta \wedge d\eta^{n-2}$.

Sasakian manifold (M, η, ξ, g, J)

- $\exists\, \xi \in \mathfrak{X}(M)$ Killing, s.t. $J(X) = -\nabla_X \xi \in \mathrm{End}(TM)$ satisfies:

$$(\nabla_X J)(Y) = g(X, Y)\xi - g(\xi, Y)X$$

- $\exists\, \xi \in \mathfrak{X}(M)$ Killing, s.t. curvature tensor R satisfies:

$$R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi := -(\nabla_X J)(Y)$$

- The metric cone $(\mathbb{R}_+ \times M, dr^2 + r^2 \cdot g)$ is Kähler.

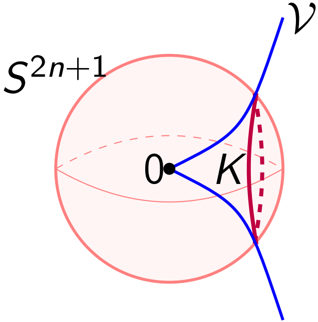
cCY links of hypersurface singularities

[Milnor'1969]

$\mathcal{V}^n \subset \mathbb{C}^{n+1}$ complex analytic, $\mathrm{sing}(\mathcal{V}) \cap B_\varepsilon(0) = \{0\}$; $S^{2n+1} := \partial B_\varepsilon(0)$.

$$K^{2n-1} := \mathcal{V} \cap S^{2n+1}$$

is a $(n-2)$-connected smooth manifold, $\dim_{\mathbb{R}} K = 2n - 1$.



Things named after Milnor

Milnor . . .

When $\mathcal{V} = (f)$, for some $f : \mathbb{C}^{n+1} \to \mathbb{C}$ holomorphic, $f(0) = 0$,

$$\phi : S^{2n+1} \setminus K \xrightarrow{\frac{f}{|f|}} S^1 \qquad \text{. . . fibration}$$

$$F := \phi^{-1}(t) \qquad \text{. . . fibre}$$

$$F \simeq S^n \vee \overset{\mu}{\ldots} \vee S^n \qquad \text{. . . number}$$

$$\mu = \dim_{\mathbb{C}} \frac{\mathcal{O}_0(\mathbb{C}^{n+1})}{J_f} = \operatorname{rk} H_n(F) \qquad \text{. . . algebra}$$

$$\boxed{\chi(F) = 1 + (-1)^n \mu}$$

The Milnor fibration ϕ is an open book decomposition with base locus

$$\partial \overline{F} = K.$$

Weighted projective CY

Weighted homogeneous polynomials

Given $w = (w_0, \dots, w_n) \in \mathrm{Q}^{n+1}$, the $\mathbb{C}^*(w)$–action on $\mathbb{C}^{n+1}$ is

$$(z_0, \dots, z_n) \mapsto (\lambda; w) \cdot (z) := (\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n)$$

and $f \in \mathbb{C}[z_0, \dots, z_n]$ is *w-homogeneous of degree d* if

$$f\left((\lambda; w) \cdot z\right) = \lambda^d f(z).$$

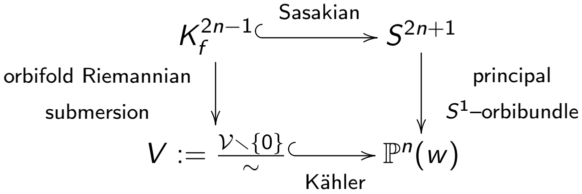
[Milnor-Orlik'70] $\boxed{\mu = \left(\frac{d}{w_0} - 1\right) \dots \left(\frac{d}{w_n} - 1\right)}$

Calabi-Yau links

K_f is a *(contact) Calabi-Yau link* if $d = \sum w_i$.

Sasakian structure of CY links

[Abe'1977]



Contact Calabi-Yau structure (cCY)

Nowhere vanishing global form $\Upsilon \in \Omega^{n,0}(M)$ s.t.

$$\Upsilon \wedge \bar{\Upsilon} = c_n \omega^n, \quad \mathrm{d}\Upsilon = 0,$$

where $c_n = (-1)^{n(n+1)/2}(\mathrm{i})^n$ and $\omega := \mathrm{d}\theta \in \Omega^{1,1}(M)$; we denote

$$\mathrm{Re}\,\Upsilon := \frac{\Upsilon + \bar{\Upsilon}}{2} \quad \text{and} \quad \mathrm{Im}\,\Upsilon := \frac{\Upsilon - \bar{\Upsilon}}{2\mathrm{i}}.$$

[HV'2015] Every CY link K_f admits a $S^1(w)$-invariant cCY structure.

G_2-structures on CY links

Fix henceforth $n = 4$, so the link K_f is a 2–connected 7-manifold:

$$\begin{array}{ccc} K_f^7 & \hookrightarrow & S^9 \\ {\scriptstyle S^1}\downarrow & & \downarrow \\ V^3 & \hookrightarrow & \mathbb{P}^4(w) \end{array}$$

[Habib–Vezzoni'2015, Calvo-Andrade–Rodríguez–S.'2020]

Every 7-dimensional Calabi-Yau link K_f with cCY structure (θ, Φ, Υ) admits the $S^1(w)$–invariant co-calibrated G_2–structure

$$\begin{array}{rcll} \varphi & := & \theta \wedge \omega + \mathrm{Im}\,\Upsilon, & \mathrm{d}\varphi = \omega \wedge \omega \\ \psi & = & *\varphi = \frac{1}{2}\omega \wedge \omega + \theta \wedge \mathrm{Re}\,\Upsilon, & \mathrm{d}\psi = 0 \end{array}.$$

Examples of Calabi-Yau links

degree	weights	polynomial	ν
75	(10,12,13,15,25)	$z_0^5 z_4 + z_1^5 z_3 + z_2^5 z_0 + z_3^5 + z_4^3$	1
135	(1,18,32,39,45)	$z_0^{135} + z_1^5 z_4 + z_2^3 z_3 + z_3^3 z_1 + z_4^3$	3
36	(18, 12, 4, 1, 1)	$z_0^2 + z_1^3 + z_2^9 + z_3^{36} + z_4^{36}$	5
81	(3,7,18,26,27)	$z_0^{27} + z_1^9 z_2 + z_2^3 z_4 + z_3^3 z_0 + z_4$	7
45	(3,5,8,14,15)	$z_0^{15} + z_1^9 + z_2^5 z_1 + z_3^3 z_0 + z_4^3$	9
45	(4,7,9,10,15)	$z_0^9 z_2 + z_2^5 + z_1^5 z_3 + z_3^3 z_4 + z_4^3$	11
75	(5,8,12,15,35)	$z_0^{15} + z_1^5 z_4 + z_2^5 z_3 + z_3^5 + z_4^2 z_0$	13
180	(90, 60, 20, 9, 1)	$z_0^2 + z_1^3 + z_2^9 + z_3^{20} + z_4^{180}$	15
45	(15, 15, 5, 9, 1)	$z_0^3 + z_1^3 + z_2^9 + z_3^5 + z_4^{45}$	17
16	(4,8,2,1,1)	$z_0^2 z_1 + z_1^2 + z_2^4 z_1 + z_3^{16} + z_4^{16} + z_2^8$	19
81	(2,9,19,24,27)	$z_0^{27} z_4 + z_2^3 z_3 + z_3^3 z_1 + z_1^9 + z_4^3$	21
24	(12, 8, 2, 1, 1)	$z_0^2 + z_1^3 + z_2^{12} + z_3^{24} + z_4^{24}$	23
1806	(42, 258, 903, 602, 1)	$z_0^{43} + z_1^7 + z_2^2 + z_3^3 + z_4^{1806}$	25
51	(2,6,9,17,17)	$z_0^{17} z_4 + z_1^7 z_2 + z_2^5 z_1 + z_3^3 + z_4^3$	29
93	(3,8,21,30,31)	$z_0^{31} + z_1^9 z_2 + z_2^3 z_3 + z_3^3 z_0 + z_4^3$	31
63	(3,4,14,21,21)	$z_0^{21} + z_1^{15} z_0 + z_2^3 z_3 + z_3^3 + z_4^3$	33
103	(1,16,23,29,34)	$z_0^{103} + z_1^5 z_2 + z_2^3 z_4 + z_3^3 z_1 + z_4^3 z_0$	37
135	(5,6,14,45,65)	$z_0^{27} + z_4^2 z_0 + z_1^{15} z_3 + z_3^3 + z_2^5 z_4$	39
60	(30, 20, 5, 4, 1)	$z_0^2 + z_1^3 + z_2^{12} + z_3^{15} + z_4^{60}$	41
55	(4,4,11,17,19)	$z_0^{11} z_2 + z_1^9 z_4 + z_2^5 + z_3^3 z_1 + z_4^2 z_3$	43
135	(1,21,30,38,45)	$z_0^{135} + z_1^5 z_2 + z_2^3 z_4 + z_3^3 z_1 + z_4^3$	45
45	(5, 5, 9, 11, 12)	$z_0^9 + z_1^8 z_0 + z_2^5 + z_4^3 z_2 + z_3^3 z_4$	47

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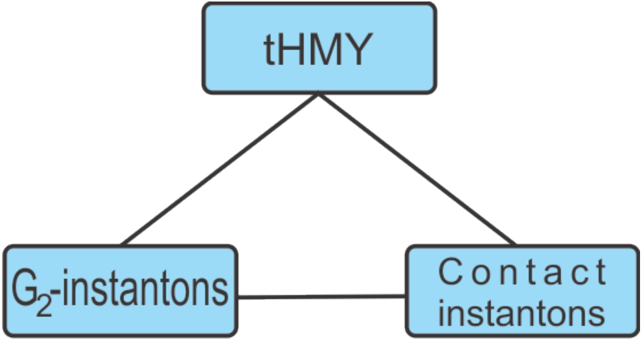
tHMY
G2-instantons
Contact
instantons

Three natural notions of instanton

- SD contact instantons (SDCI).
- **Transverse Hermitian Yang-Mills** (tHYM): $\hat{F}_A := (F_A, \omega) = 0$ and $F_A^{0,2} = 0$. The concept of **Chern connection** also extends naturally.
- G_2-**instantons**: For M cCY(2) ▸ InsEquation its natural G_2-structure $\varphi := \sigma + \mathrm{Im}(\Upsilon)$,

Theorem (Portilla-S.'2019)

$\mathcal{E} \to M$ Sasakian holomorphic on a cCY manifold; then:

(i) A Chern connection is tHYM $\iff$ it is a G_2-instanton.

(ii) Every contact instanton is a G_2-instanton.

(iii) A Chern connection is a G_2-instanton $\iff$ it is a contact instanton.

In particular, among Chern connections the three notions are equivalent.

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Ingredient: cCY^7 with small fibres

Let $(V^3, g_V, \omega, \Upsilon)$ be a CY 3-orbifold, with

$$\mathrm{vol}_V = \frac{\omega^3}{3!} = \frac{\mathrm{Re}\,\Upsilon \wedge \mathrm{Im}\,\Upsilon}{4}. \tag{5}$$

Suppose that the total space of $\pi : K \to V$ is a cCY^7, i.e. K is a S^1-(orbi)bundle, with connection 1-form η, such that $d\eta = \omega$.

S^1-invariant G_2-structure on K

Fixing $\varepsilon = \varepsilon(\alpha') > 0$, define

$$\varphi_\varepsilon = \varepsilon\eta \wedge \omega + \mathrm{Re}\,\Upsilon, \tag{6}$$

$$\psi_\varepsilon = \frac{1}{2}\omega^2 - \varepsilon\eta \wedge \mathrm{Im}\,\Upsilon, \tag{7}$$

NB.:

$$g_\varepsilon = \varepsilon^2 \eta \otimes \eta + g_V \quad \text{and} \quad \mathrm{vol}_\varepsilon = \varepsilon\eta \wedge \mathrm{vol}_V. \tag{8}$$

Main theorem: existence of heterotic G_2 data on cCY^7

Theorem

Let $(K^7, \eta, \xi, J, \Upsilon)$ *be a* cCY^7*, fibering by* $\pi : K^7 \to V$ *over the Calabi–Yau 3-fold* $(V, g_V, \omega, J, \Upsilon)$*, and let* $E := \pi^* TV \to K$*.*
Given any $\alpha' > 0$*, there exist* $k(\alpha'), \varepsilon(\alpha') > 0$ *and* $m, \delta \in \mathbb{R}$ *such that:*

1. *The* G_2*-structure* (6) *is coclosed and satisfies the torsion conditions* (3)*, with* $\lambda = \frac{\varepsilon}{2}$*,* $\mu \in \mathbb{R}$*,* $H_\varepsilon = -\varepsilon^2 \eta \wedge \omega + \varepsilon \operatorname{Re} \Upsilon$*.*
2. $A := \pi^* \Gamma_V$ *is a* G_2*-instanton on* E*, wrt* (7)*.*
3. *There exists a connection* $\theta := \theta^{\delta,k}_{\varepsilon,m}$ *on* TK*, which is a* G_2*-instanton to order* $O(\alpha')^2$*, wrt* (7)*.*
4. $(H_\varepsilon, A, \theta)$ *satisfy the heterotic Bianchi identity* (4)*:*
$$\mathrm{d}H_\varepsilon = \frac{\alpha'}{4}(\operatorname{tr} F_A^2 - \operatorname{tr} R_\theta^2). \tag{9}$$
5. $\lim_{\alpha' \to 0} \varepsilon(\alpha') = 0$ *and* $\lim_{\alpha' \to 0} k(\alpha') = \infty$*.*

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Torsion forms and flux of φ_ε

It is easy to check that

$$\mathrm{d}\varphi_\varepsilon = \varepsilon\omega^2 \quad \text{and} \quad \mathrm{d}\psi_\varepsilon = 0. \tag{10}$$

Torsion forms and flux of φ_ε

$$\begin{aligned} \tau_0 &= \frac{6}{7}\varepsilon, \quad \tau_1 = 0, \\ \tau_2 &= 0, \qquad \tau_3 = \frac{8}{7}\varepsilon^2\eta\wedge\omega - \frac{6}{7}\varepsilon\,\mathrm{Re}\,\Upsilon. \end{aligned} .$$

$$H_\varepsilon = -\varepsilon^2\eta\wedge\omega + \varepsilon\,\mathrm{Re}\,\Upsilon. \tag{11}$$

In particular, the prescribed Chern-Simons defect is

$$\mathrm{d}H_\varepsilon = -\varepsilon^2\omega^2. \tag{12}$$

The G_2-instanton A on $E := \pi^* TV$

Since the Levi-Civita connection of the Calabi–Yau (V, g_V) on TV is HYM:

Lemma

Let $E = \pi^* TV$ be the pullback via the projection $\pi : K \to V$. Then $A := \pi^* \Gamma_V$ is a G_2-instanton on $E = \pi^* TV$, with holonomy contained in $SU(3)$.

Local orthonormal coframe

Definition

Given $\varepsilon > 0$, choose the local Sasakian real orthonormal coframe on K:

$$e_0 = \varepsilon\eta, \quad e_1, \quad e_2, \quad e_3, \quad Je_1, \quad Je_2, \quad Je_3, \tag{13}$$

where J is the transverse complex structure acting on 1-forms, such that

$$\omega = e_1 \wedge Je_1 + e_2 \wedge Je_2 + e_3 \wedge Je_3, \tag{14}$$

$$\Upsilon = (e_1 + iJe_1) \wedge (e_2 + iJe_2) \wedge (e_3 + iJe_3). \tag{15}$$

The Levi-Civita connection θ_ε as a perturbed G_2-instanton

Structure equations of the coframe (13) on K

Letting $I := (\delta_{ij})$ and $e := (e_1\ e_2\ e_3)^{\mathrm{T}}$, the Levi-Civita connection θ_ε of the metric g_ε in (8) is given locally by

$$d\begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} = -\underbrace{\begin{pmatrix} 0 & \frac{\varepsilon}{2}Je^{\mathrm{T}} & -\frac{\varepsilon}{2}e^{\mathrm{T}} \\ -\frac{\varepsilon}{2}Je & a & b-\frac{\varepsilon}{2}e_0 I \\ \frac{\varepsilon}{2}e & -b+\frac{\varepsilon}{2}e_0 I & a \end{pmatrix}}_{\theta_\varepsilon = A+\frac{\varepsilon}{2}B} \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix}, \qquad (16)$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & Je^{\mathrm{T}} & -e^{\mathrm{T}} \\ -Je & 0 & -e_0 I \\ e & e_0 I & 0 \end{pmatrix}, \qquad (17)$$

NB.: Since A has holonomy in $\mathrm{SU}(3)$, get local 1-forms a_{ij}, b_{ij}, $1 \leq i,j \leq 3$, st. $\quad a_{ji} = -a_{ij}, \quad b_{ji} = b_{ij}, \quad \sum_{i=1}^{3} b_{ii} = 0.$

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Squashed Levi-Civita connection θ_ε^k on TK

Definition

For each $0 \neq k \in \mathbb{R}$, let θ_ε^k be the connection on TK given, in the local coframe, by

$$\theta_\varepsilon^k := A + \frac{k\varepsilon}{2} B,$$

1. Since $K \to V$ is a non-trivial S^1-bundle, we require $k \neq 0$.
2. We may view θ_ε^k as a metric connection on K, with torsion $(1-k)\varepsilon\omega \otimes e_0$:

$$d \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} = -\theta_\varepsilon^k \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} + (1-k)\varepsilon \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}.$$

Since $k \neq 0$, θ_ε^k is essentially the LC of $g_{k\varepsilon}$, but in the metric g_ε, θ_ε^k appears as a "squashing" of the LC connection θ_ε of g_ε.

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Connections with torsion induced by the flux

Let $\mathcal{I} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -I_3 \\ 0 & I_3 & 0 \end{pmatrix}$ and $\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}$.

Proposition

In the local Sasakian coframe, let

$$C := \begin{pmatrix} 0 & Je^{\mathrm{T}} & -e^{\mathrm{T}} \\ -Je & -[e] & e_0 I + [Je] \\ e & -e_0 I + [Je] & [e] \end{pmatrix} = \begin{pmatrix} 0 & Je^{\mathrm{T}} & -e^{\mathrm{T}} \\ -Je & -[e] & [Je] \\ e & [Je] & [e] \end{pmatrix} - e_0 \mathcal{I}. \tag{18}$$

Raising an index on the 3-form H_ε, we view it as a vector-valued 2-form:

$$H_\varepsilon = \frac{\varepsilon}{2} \begin{pmatrix} 0 & Je^{\mathrm{T}} & -e^{\mathrm{T}} \\ -Je & -[e] & e_0 I + [Je] \\ e & -e_0 I + [Je] & [e] \end{pmatrix} \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} = \frac{\varepsilon}{2} C \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix}. \tag{19}$$

Bismut, Hull and beyond

Corollary

In the terms of Definition 5 and Proposition 3.1, let $\tau_\varepsilon := \varepsilon C$*; then each local matrix*

$$\theta_\varepsilon^{\delta,k} = \theta_\varepsilon^k + \frac{k\delta}{2}\tau_\varepsilon = A + \frac{k\varepsilon}{2}B + \frac{k\varepsilon\delta}{2}C, \quad \text{for} \quad k \neq 0 \text{ and } \delta \in \mathbb{R}, \tag{20}$$

defines a connection on TK*, with torsion*

$$H_\varepsilon^{\delta,k} = (1-k)\varepsilon\omega \otimes e_0 + k\delta H_\varepsilon. \tag{21}$$

Alas!

$\theta_\varepsilon^{\delta,k}$ is never a G_2-instanton on TK.

An extra twist

Proposition

In the local Sasakian coframe, define a connection $\theta^{\delta,k}_{\varepsilon,m}$ on TK by

$$\theta^{\delta,k}_{\varepsilon,m} = \theta^{\delta,k}_{\varepsilon} + \frac{km\varepsilon}{2} e_0 \mathcal{I}. \tag{22}$$

Hooray!

$\theta^{\delta,k}_{\varepsilon,m}$ is an "approximate" G_2-instanton on TK whenever these are $O(\alpha')^2$:

$$\lambda_1 := \frac{k\varepsilon^2\big(6(1-\delta+m)+k(1-\delta)(1+3\delta)\big)}{4},$$

$$\lambda_2 := \frac{k^2\varepsilon^2}{4}(1+m-5\delta)(1+\delta),$$

$$\lambda_3 := \frac{k^2\varepsilon^2}{4}(\delta^2-2(2+m)\delta-1).$$

Asymptotics for small $\alpha' > 0$

The heterotic Bianchi identity for $\theta = \theta_{\varepsilon,m}^{\delta,k}$ and φ_ε is:

$$\mathrm{d}H_\varepsilon = -\varepsilon^2\omega^2 = \frac{\alpha'}{4}(\mathrm{tr}\, F_A^2 - \mathrm{tr}\, R_\theta^2). \quad (23)$$

Proposition

There is an approximate solution to the heterotic G_2 system if and only if

$$\lambda_0 := k^2\varepsilon^2\left(k^2\delta^2(1+\delta)^2 + (1-\delta+m)\left(k(4\delta^2-(1+\delta)^2)-3\right)\right) > 0 \quad (24)$$

is large so that

$$\alpha' = \frac{8}{\lambda_0} > 0 \quad (25)$$

is small and the terms in the G_2-instanton condition are all $O(\alpha')^2$.

Many solutions! Eg. $1 - \delta + m = 0$ and $\delta \neq 0, -1$:

$$\begin{aligned} \alpha' &= \tfrac{8}{\delta^2(1+\delta)^2}\tfrac{1}{\varepsilon^2 k^4}, & \lambda_2 &= -\delta(1+\delta)k^2\varepsilon^2, \\ \lambda_1 &= \tfrac{(1-\delta)(1+3\delta)}{4}k^2\varepsilon^2, & \lambda_3 &= -\tfrac{(\delta+1)^2}{4}k^2\varepsilon^2. \end{aligned}$$

In order to have $k^2\varepsilon^2 = O(\alpha')^2$, we may take

$$k^2 = \frac{1}{(\alpha')^3} \quad \text{and} \quad \varepsilon^2 = \frac{8}{\delta^2(1+\delta)^2}(\alpha')^5,$$

with $\delta \neq 0, -1 \quad$ and $\quad m = \delta - 1$,
which is physically meaningful with $\varepsilon \ll 1$ and $k \gg 1$.

Many solutions! Eg. $\delta = 0$ and $(1+m)(k+3) < 0$:

$$\alpha' = -\frac{8}{(1+m)(1+\frac{3}{k})}\frac{1}{\varepsilon^2 k^3}, \qquad \lambda_2 = \frac{1+m}{4}k^2\varepsilon^2,$$
$$\lambda_1 = \frac{\left(1+\frac{6(1+m)}{k}\right)}{4}k^2\varepsilon^2, \qquad \lambda_3 = -\frac{1}{4}k^2\varepsilon^2.$$

In order to have $k^2\varepsilon^2 = O(\alpha')^2$, we may take

$$k = \frac{1}{(\alpha')^3} \quad \text{and} \quad \varepsilon^2 = \frac{8}{(1+m)(1+3(\alpha')^3)}(\alpha')^8, \quad \text{with} \quad m < -1,$$

which is physically meaningful with $\varepsilon \ll 1$ and $k \gg 1$.

Many solutions! Eg. $\delta = -1$ and $(2+m)(4k-3) > 0$

$$\begin{aligned} \alpha' &= \tfrac{8}{(2+m)(4-\frac{3}{k})}\tfrac{1}{\varepsilon^2 k^3}, & \lambda_2 &= 0, \\ \lambda_1 &= \left(\tfrac{3(2+m)}{2k} - 1\right) k^2\varepsilon^2, & \lambda_3 &= -\tfrac{2+m}{2} k^2 \varepsilon^2. \end{aligned}$$

In order to have $k\varepsilon^2 = O(\alpha')^2$ and $k^2\varepsilon^2 = O(\alpha')^2$, we may take

$$k = \frac{1}{(\alpha')^3} \quad \text{and} \quad \varepsilon^2 = \frac{8}{(2+m)(4-3(\alpha')^3)}(\alpha')^8, \quad \text{with} \quad m > -2,$$

which is physically meaningful with $\varepsilon \ll 1$ and $k \gg 1$.

1. Several other solution regimes are possible, in particular one may adjust the choices of m and δ to the string scale α' itself.
2. The asymptotics of $\varepsilon(\alpha')$ and $k(\alpha')$ as $\alpha' \to 0$ are a *consequence* of the heterotic Bianchi identity and the G_2-instanton condition 'up to $O(\alpha')^2$ terms', and therefore *not a choice* imposed on the Ansatz.

Merci!