Statistical hyperbolicity of Teichmüller spaces

Vaibhav Gadre
University of Glasgow

June 18, 2020

Joint with
Luke Jeffreys (Glasgow/ Bristol)
Aitor Azemar (Glasgow)

Sprawl

Let X be a metric space and $x \in X$ a base-point.

Let μ_R be probability measures supported on spheres $S(x, R)$ centred at x and with radius R.

Duchin-Lelievre-Mooney:

$$E(X, x, \mu) = \lim_{R \to \infty} \frac{1}{R} \int_{S(x,R) \times S(x,R)} d(y, z)\, d\mu_R(y)\, d\mu_R(z).$$

In other words, "$E(X, x, \mu)$ is the normalised average distance between pairs of random points on spheres as radius goes to infinity".

Properties of the sprawl

Assuming that the limit exists

- $E(X, x, \mu) \leqslant 2$,
- $E(X, x, \mu)$ is base-point dependent,
- $E(X, x, \mu)$ is not a quasi-isometry invariant,
- $E(X, x, \mu)$ is dependent on the family of measures μ_R.

Statistical hyperbolicity

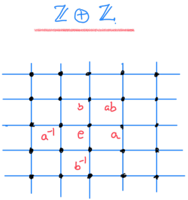
A space X is *statistically hyperbolic* with respect to the measures μ_R if $E(X, x, \mu)$ is base-point independent and equals 2.

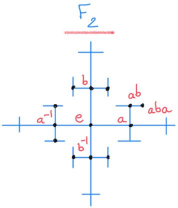
Without homogeneity, easy to construct Gromov hyperbolic graphs that are not statistically hyperbolic.

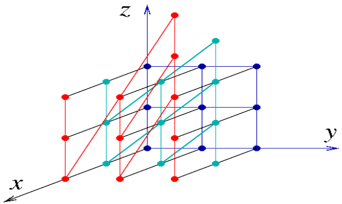
- By adding edges to increase branching along an axis in a 4-regular tree, $E(X, x, \mu_{\text{count}})$ can be arranged to take any value in $[0, 2]$.

Cayley Graphs

Let G be a finitely gen. group with a symmetric generating set $\mathcal{A}$. Let $\mathrm{Cay}(G, \mathcal{A})$ be the Cayley graph; vertices are group elements, edge between g and h if $g = ha$ for some $a \in S$.







Sprawls of Cayley graphs and statistical hyperbolicity

Let μ_R^{count} be the counting measure on the radius R sphere in the Cayley graph $\text{Cay}(G, \mathcal{A})$.

Duchin-Lelievre-Mooney: For any finite sym. gen. set $\mathcal{A}$ the sprawl $E(\text{Cay}(\mathbb{Z}^d, \mathcal{A}), \mu^{\text{count}}) < 2$.

Duchin-Lelievre-Mooney: If G is a (non-elementary) Gromov hyperbolic group then $\text{Cay}(G, \mathcal{A})$ is statistically hyperbolic for any finite sym. gen. set $\mathcal{A}$.

The direct product $F_2 \times \mathbb{Z}$ is statistically hyperbolic.

Osborne-Yang: Relatively hyperbolic groups are statistically hyperbolic for any finite generating set.

Teichmüller spaces

Let S be an oriented surface of finite type.

- Teichmüller space of S = space of marked conformal structures on S.
- Mapping class group of $S = \mathrm{Mod}(S)$ = orientation preserving diffeos of S mod isotopies.
- Moduli Space = $\mathrm{Teich}(S)/\,\mathrm{Mod}(S)$.

Teichmüller metric

The diagonal part of the $SL(2, \mathbb{R})$ action: For any marked conformal structures X, Y on the surface S

- there exist quadratic differentials q_X and q_Y and
- the "extremal" quasi-conformal map $X \to Y$ is affine in the flat structure of q_X stretching the horizontal and shrinking the vertical to give q_Y.

Teichmüller metric is given by

$$d(X, Y) = \frac{1}{2} \log(\text{stretch factor}).$$

Exceptional moduli: The modular surface

Realising a 2-torus as $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$ with $\mathrm{Im}(\tau) > 0$ defines a marked conformal structure on it.

Mapping class group in this case is the modular group $SL(2, \mathbb{Z})$.

The quotient $\mathbb{H}/SL(2, \mathbb{Z})$ is the moduli space of tori.

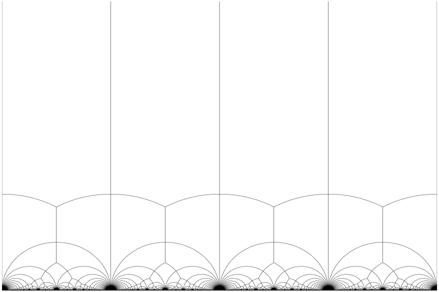


Figure: From Cheritat's gallery

Properties of Teichmüller metric

- Mod(S)-invariant
- Finsler and inhomogeneous
  - Thick-thin decomposition; thick and thin points have different local geometry.
- Not Gromov hyperbolic
  - Thick part has aspects of negative curvature.
  - **Minsky:** Thin part has product structure.

Dowdall-Duchin-Masur: For many "Lebesgue class" measures, Teichmüller space with the Teichmüller metric is statistically hyperbolic.

Random walks

Let μ be a probability distribution on a fin. gen. group.

A sample path $w_n = g_1 \cdots g_n$, with each g_k sampled by μ.

Example: Brownian motion on $SL(2, \mathbb{R})$ and its convergence to the boundary $S^1 = \mathbb{H}^2$.

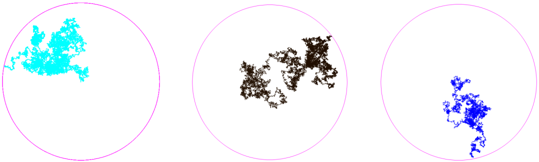


Figure: Applet from P. Storm's webpage at UPenn

Random walks on non-uniform lattices in $SL(2, \mathbb{R})$

Random walks on non-uniform lattices also give convergence to S^1.

Example: $SL(2, \mathbb{Z})$ with generators R, L and a non-backtracking nearest neighbour random walk.

- $SL(2, \mathbb{Z})$ is quasi-isometric to the dual of the Farey graph.
- A typical non-backtracking sample path $\omega = (w_n)$ gives a sequence $R^{a_1} L^{a_2} R^{a_3} \cdots$ and converges to the point with continued fraction $[a_1, a_2, a_3, \cdots]$.

Guivarc'h-Lejan, Blachere-Haissinsky-Mathieu, Deroin-Kleptsyn-Navas, G-Maher-Tiozzo:
If μ has finite word-metric first moment, its stationary measure on S^1 is singular.

Random walks on mapping class groups

- **Kaimanovich-Masur:** For any base-point x, the typical sample path $\omega = (w_n)$ gives a sequence $w_n x$ that converges to a point in the Thurston boundary.
- **G, G-Maher-Tiozzo:** If μ is non-elementary and has finite first moment with respect to a word metric, then its stationary measure is singular.

Statistical hyperbolicity

Stationary measures are supported on uniquely ergodic projective measured foliations, so pullback to the unit tangent sphere.

Azemar-G-Jeffreys: If μ is non-elementary and has finite first moment with respect to Teichmüller metric, then Teichmüller space is statistically hyperbolic with respect to its stationary measure.

Eskin-Mirzakhani-Rafi: There exists μ with finite first Teichmüller moment such that its stationary measure is absolutely continuous.

Separation, thickness and triangles

Dowdall-Duchin-Masur: Strategy of proof as below.

- Step 1: Typical pairs of Teichmüller rays separate.
- Step 2: Typical rays spend a definite proportion of time in the thick part $\Rightarrow$ for typical pairs (γ, γ') of rays, the triangle with sides γ and γ' and the segment $[\gamma_R, \gamma'_R]$ is "thin".

Recurrence and drift for random walks

- **Kaimianovich-Masur:** Typical sample paths track geodesics recurring infinitely often to a fixed neighbourhood of the tracked geodesic.
- **Maher:** Typical sample paths have linear drift.

Azemar-G-Jeffreys: Quantitative thickness of tracked geodesics from recurrence and drift.

Hyperbolicity of the curve complex and separation

The curve complex $\mathcal{C}(S)$ is the graph with vertices isotopy classes of simple closed curves on S with edges between disjoint curves.

Masur-Minsky:

- $\mathcal{C}(S)$ is Gromov-hyperbolic.
- The systole projection from Teichmüller space to $\mathcal{C}(S)$ takes Teichmüller geodesics to un-parameterised quasi-geodesics.
- Projections of thick Teichmüller geodesic segments make definite progress.

Azemar-G-Jeffreys: Long fellow travelling of geodesics $\Rightarrow$ nesting into a "small" shadow $\Rightarrow$ separation for typical pairs of geodesics.

Future directions

Masur-Minsky: Machinery of hierarchies for mapping class groups.

Behrstock-Hagen-Sisto: Hierarchically hyperbolic spaces/ groups modelled on the hierarchy machinery.

- Statistical hyperbolicity for hierarchically hyperbolic spaces?

Thank you for your time!