

# Gravitating vortices with positive curvature

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# **Abelian vortices**

The (abelian) vortex equation for a Hermitian metric  $h$  on a line bundle  $L$  over a Riemann surface  $\Sigma$  with section  $\phi \in H^0(\Sigma, L)$

$$*F_h - \frac{i}{2}(|\phi|_h^2 - \tau) = 0$$

is a generalization of the equations on  $\mathbb{R}^2$  which were introduced in 1950 by Ginzburg and Landau in the theory of superconductivity.

The equations, often called Bogomol'nyi equations, depend on a choice of background metric  $g$  on  $\Sigma$  and a *symmetry breaking parameter*

$$0 < \tau \in \mathbb{R}.$$

In this physical setup, the Chern connection  $A_h = h^{-1}\partial h$  represents the electromagnetic field and  $\phi$  is the electron wave function density.

*Abelian vortices have been extensively studied in the mathematics literature after the seminal work of Jaffe and Taubes on the Euclidean plane, and Witten on the 2-dimensional Minkowski spacetime.*

Assuming that  $\Sigma$  is compact, the existence problem for abelian vortices was solved independently by Noguchi, Bradlow, and Garcia-Prada:

Theorem (Noguchi '87, Bradlow '90, Garcia-Prada '93)

$L$  holomorphic line bundle over a compact Riemann surface  $\Sigma$  with Kähler metric  $g$ , with section  $0 \neq \phi \in H^0(\Sigma, L)$ . Fix  $0 < \tau \in \mathbb{R}$ . Then,

$$i\Lambda_g F_h + \frac{1}{2}(|\phi|_h^2 - \tau) = 0$$

admits a solution  $h$  if and only if (where  $N := \int_{\Sigma} c_1(L)$ )

$$\frac{4\pi N}{\text{Vol}_g} < \tau.$$

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**Proof:** Assume that  $h$  is a solution. Integrating the equation

$$iF_h + \frac{1}{2}(|\phi|_h^2 - \tau)\omega = 0 \quad (1)$$

we obtain ( $\phi \neq 0$ )

$$4\pi N + (\|\phi\|_{L^2}^2 - \tau)\text{Vol}_g = 0 \quad \implies \quad 4\pi N/\text{Vol}_g < \tau.$$

Assume now  $4\pi N/\text{Vol}_g < \tau$ . Consider the  $SU(2)$ -equivariant bundle on  $X = \Sigma \times \mathbb{P}^1$

$$0 \rightarrow p^*L \rightarrow E \rightarrow q^*O_{\mathbb{P}^1}(2) \rightarrow 0$$

determined by

$$\phi \in H^0(\Sigma, L) \cong H^1(X, p^*L \otimes q^*O_{\mathbb{P}^1}(-2)).$$

Then, for the Kähler form  $\omega_\tau = p^*\omega + \frac{4}{\tau}q^*\omega_{FS}$ , Garcia-Prada proves that

$$E \text{ is slope } [\omega_\tau] \text{ - stable} \Leftrightarrow 4\pi N < \tau \text{Vol}_g$$

$$E \text{ admits } \omega_\tau \text{ - Hermite-Einstein metric} \Leftrightarrow L \text{ admits a solution of (1)}$$

The proof follows from the Donaldson-Uhlenbeck-Yau's Theorem. □

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Bradlow–Garcia-Prada–Noguchi's Theorem identifies the moduli space of abelian vortices on  $\Sigma$  with the symmetric product

$$\mathcal{M} = S^N \Sigma.$$

This moduli space carries an interesting Kähler metric, obtained by infinite-dimensional Kähler reduction, extensively studied in the mathematical physics literature.

*'We assume vortices have no back-reaction on the metric. They are not gravitating. Some vortices - cosmic strings - have a gravitational effect.'*

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## **Gravitating vortices**

Motivated by the *Kähler-Yang-Mills equations* in higher dimensions, jointly with Álvarez-Cónsul and García-Prada we introduced a notion of vortex on a Riemann surface with back-reaction of the metric.

- *Gravitating vortices, cosmic strings, and the Kähler-Yang-Mills equations*, **Comm. Math. Phys.** 351 (2017) (Álvarez-Cónsul, GF, García-Prada).

### Definition (Álvarez-Cónsul, GF, García-Prada '17)

$L$  holomorphic line bundle over a Riemann surface  $\Sigma$ , with section  $0 \neq \phi \in H^0(\Sigma, L)$ . Fix  $0 < \tau \in \mathbb{R}$  and  $0 \leq \alpha \in \mathbb{R}$ .

The *gravitating vortex equations* for  $(g, h)$ , where  $g$  is a Kähler metric on  $\Sigma$  and  $h$  is a hermitian metric on  $L$ , are given by

$$\begin{aligned} i\Lambda_g F_h + \frac{1}{2}(|\phi|_h^2 - \tau) &= 0, \\ S_g + \alpha(\Delta_g + \tau)(|\phi|_h^2 - \tau) &= c. \end{aligned} \tag{2}$$

The constant  $c \in \mathbb{R}$  is topological, given explicitly by

$$c = \frac{2\pi(\chi(\Sigma) - 2\alpha\tau N)}{\text{Vol}_g}. \tag{3}$$

Consider the  $SU(2)$ -equivariant bundle on  $X = \Sigma \times \mathbb{P}^1$  determined by  $\phi \in H^0(\Sigma, L)$ :

$$0 \rightarrow p^*L \rightarrow E \rightarrow q^*O_{\mathbb{P}^1}(2) \rightarrow 0.$$

### Proposition (Álvarez-Cónsul, GF, García-Prada '17)

Fix  $0 < \tau \in \mathbb{R}$  and  $0 \leq \alpha \in \mathbb{R}$ . The gravitating vortex equations admit a solution with Kähler form  $\omega$  if and only if  $(X, E)$  admits a solution  $(g_X, H)$  of the Kähler-Yang-Mills equations

$$\begin{aligned} i\Lambda F_H &= \lambda \text{Id}, \\ S_{g_X} - \alpha \Lambda^2 \text{tr} F_H \wedge F_H &= c_X. \end{aligned} \tag{4}$$

with Kähler form  $\omega_X = p^*\omega + \frac{4}{\tau}q^*\omega_{FS}$ .

**Upshot:** the gravitating vortex equations inherit a **moment map interpretation** from the Kähler-Yang-Mills equations.

- *Coupled equations for Kähler metrics and Yang-Mills connections, Geometry and Topology 17 (2013) (Álvarez-Cónsul, GF, García-Prada).*

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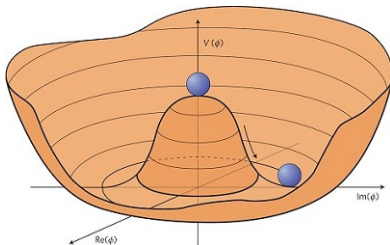
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When  $c = 0$ , that is,  $\Sigma = \mathbb{P}^1$  and  $\alpha = 1/\tau N$ , we find the *Einstein-Bogomol'nyi equations*, which describe phase transitions in the early universe in a current cosmological theory:

$$\begin{aligned} i\Lambda_g F_h + \frac{1}{2}(|\phi|_h^2 - \tau) &= 0, \\ S_g + \alpha(\Delta_g + \tau)(|\phi|_h^2 - \tau) &= 0. \end{aligned} \tag{5}$$

The higgs field  $\phi$  decays to a minima of the potential  $V(\phi) = (|\phi|^2 - \tau)^2$  'breaking the symmetry'. The 'choice of point' in the circle  $|\phi|^2 = \tau$  may vary in space-time causing *topological defects* (cosmic strings).



- P. Ade et al., *Planck 2013 results. XXV. Searches for cosmic strings and other topological defects*, **Astronomy and Astrophysics** 571 (2014).

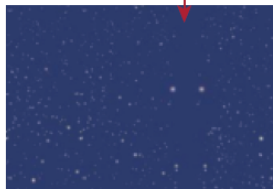
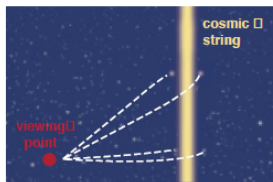
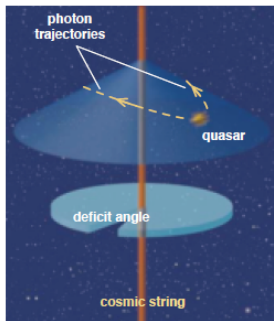


Figure 6. Cosmic strings affect surrounding spacetime by removing a small angular segment, creating a conelike geometry (*above*). Space remains flat everywhere, but a circular path around the string encompasses slightly less than 360 degrees. The deficit angle is tiny, about  $10^{-5}$  radian. To an observer, the presence of a cosmic string would be betrayed by its effect on the trajectory of passing light rays, which are deflected by an amount equal to the deficit angle. The resultant gravitational lensing reveals itself in the doubling of images of objects behind the string (*bottom panel, right*).

Figure: Comtet-Gibbons' solution '88 on  $\mathbb{R}^{1,1} \times \mathbb{C}^*$  (pure gravity):  $-dt^2 + dx^3 + |z - z_0|^{-\frac{\delta}{\pi}} dzd\bar{z}$ , deficit angle  $\delta$ .

From A. Gangui 'Superconducting strings', Am. Scientists, 2000

**Existence for  $c \leq 0$**

Take functions  $u, f \in C^\infty(\Sigma)$  and consider  $g = (1 - \Delta u)g_0$  and  $h = e^{2f}h_0$ , for a suitable background geometry  $(g_0, h_0)$ . Then, the gravitating vortex equations for  $(g, h)$  are equivalent to

$$\Delta f + \frac{1}{2}(e^{2f}|\phi|^2 - \tau)e^{4\alpha\tau f - 2\alpha e^{2f}|\phi|^2 - 2cu} = -N,$$

$$\Delta u + e^{4\alpha\tau f - 2\alpha e^{2f}|\phi|^2 - 2cu} = 1.$$



### Theorem (Álvarez-Cónsul, GF, García-Prada, Pingali '18)

Let  $\Sigma$  be a compact connected Riemann surface with  $g(\Sigma) \geq 2$ . Then, there exists a solution of the gravitating vortex equations with volume  $2\pi$  provided that

$$0 \leq \alpha \leq \frac{2g(\Sigma) - 2}{\tau(\tau - 2N)}, \quad 0 < \tau - 2N$$

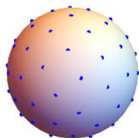
- *Gravitating vortices and the Einstein-Bogol'nyi Equations*,  
**Math. Annalen** (2020), (with Álvarez-Cónsul, García-Prada, Pingali).

Assume now that

$$c = \frac{2\pi(\chi(\Sigma) - 2\alpha\tau N)}{\text{Vol}_g} = 0.$$

Then  $\Sigma \cong \mathbb{P}^1$  and  $\alpha = 1/\tau N$ , and the *Einstein-Bogomol'nyi equations* reduce to a single PDE for a function on the two-sphere

$$\Delta f + \frac{1}{2}(e^{2f}|\phi|^2 - \tau)e^{4\alpha\tau f - 2\alpha e^{2f}|\phi|^2} = -N.$$



### Theorem (Yang '95, '97)

Let  $D = \sum_i n_i p_i$  be the effective divisor on  $\mathbb{P}^1$  corresponding to the pair  $(L, \phi)$ .

- 1 Assume that  $n_i < \frac{N}{2}$  for all  $i$ . Then, there exists  $V_0 \gg 4\pi N/\tau$  such that the Einstein-Bogomol'nyi equations admit a solution on any Kähler class such that  $\text{Vol}_g > V_0$ .
- 2 Assume that  $D = \frac{N}{2}p_0 + \frac{N}{2}p_1$ . Then, the Einstein-Bogomol'nyi equations admit a solution on any Kähler class such that  $\text{Vol}_g > 4\pi N/\tau$ . Furthermore, the solution is  $S^1$ -symmetric.

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### Yang's method:

1) Yang considers the  $\varepsilon$ -rescaled PDE (corresponding to  $(\frac{1}{\varepsilon}(1 - \Delta u)g_0, e^{2f} h_0)$ )

$$\Delta f + \frac{1}{2\varepsilon}(e^{2f}|\phi|^2 - \tau)e^{4\alpha\tau f - 2\alpha e^{2f}|\phi|^2} = -N.$$

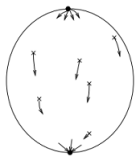
Smoothing  $\log|\phi|_{h_0}^2$ , takes sequence of supersolutions.  $C^0$ -estimate requires

$$|\phi|_{h_0}^{-2/N} \in L^p, p > 1.$$

2) Follows by reduction to ODE.

Of course, the conditions in Yang's Theorem correspond to Mumford GIT stability for the linearised  $SL(2, \mathbb{C})$ -action on  $S^N \mathbb{P}^1$

- $n_i < \frac{N}{2}$ , for every  $i \iff D \in S^N \mathbb{P}^1$  stable
- $D = \frac{N}{2} p_0 + \frac{N}{2} p_1 \iff D \in S^N \mathbb{P}^1$  strictly polystable



Theorem (Álvarez-Cónsul - GF - García-Prada - Pingali '18)

Let  $0 < \tau, \alpha \in \mathbb{R}$ . Let  $D = \sum_i n_i p_i$  be the effective divisor on  $\mathbb{P}^1$  corresponding to the pair  $(L, \phi)$ . If  $(\mathbb{P}^1, L, \phi)$  admits a solution of the gravitating vortex equations, then the following hold

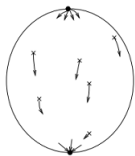
- 1  $Vol_g > 4\pi N/\tau$ ,
- 2  $D$  is polystable with respect to the  $SL(2, \mathbb{C})$ -action on  $S^N \mathbb{P}^1$ .

**Method:** convexity of  $\alpha$ -Mabuchi energy along geodesic rays combined with Futaki invariant

$$\mathcal{F}_{\alpha, \tau} = 2\alpha(\tau Vol_g - 4\pi N)(N - 2n_i).$$

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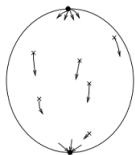
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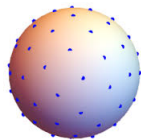
**Existence for  $c > 0$**   
(positive curvature)

Assume now that  $\alpha > 0$  and

$$c = \frac{2\pi(\chi(\Sigma) - 2\alpha\tau N)}{\text{Vol}_g} > 0.$$

Then  $\Sigma \cong \mathbb{P}^1$  and  $0 < \alpha < 1/\tau N$ .

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Let  $D = \sum_i n_i p_i$  be the effective divisor on  $\mathbb{P}^1$  corresponding to the pair  $(L, \phi)$ . Fix  $\tau > 0$  and  $\alpha$  such that  $0 < \alpha < 1/\tau N$  ( $c > 0$ ).

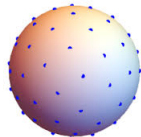
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## Theorem (GF - Pingali - Yao '20)

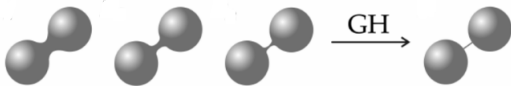
Let  $D = \sum_i n_i p_i$  be the effective divisor on  $\mathbb{P}^1$  corresponding to the pair  $(L, \phi)$ . Fix  $\tau > 0$  and  $\alpha$  such that  $0 < \alpha < 1/\tau N$  ( $c > 0$ ).

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**Idea:** apply continuity method in the parameter  $\alpha$  starting at Yang's solution for  $\alpha = 1/\tau N$  ( $c = 0$ ).

### Difficulties:

- Yang's result only holds for  $V_g \gg 4\pi N/\tau$  when  $D \neq \frac{N}{2} p_0 + \frac{N}{2} p_1$ ,
- $S^1$ -symmetry potentially obstructs openness when  $D = \frac{N}{2} p_0 + \frac{N}{2} p_1$ ,
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Let  $D = \sum_i n_i p_i$  be the effective divisor on  $\mathbb{P}^1$  corresponding to  $(L, \phi)$ . Assume that  $D$  is stable with respect to the  $SL(2, \mathbb{C})$ -action.

Then, the Einstein-Bogomol'nyi equations ( $c = 0$  and  $\alpha = 1/\tau N$ ) admit a solution on any Kähler class such that  $Vol_g > 4\pi N/\tau$ .

**Idea:** start with Yang's solution at large volume  $Vol_g \gg 4\pi N/\tau$  and apply continuity method with parameter  $Vol_g$  (requires Cheeger-Gromov theory + Lebrun-Simanca type argument for the case  $D_0 = \frac{N}{2}p_0 + \frac{N}{2}p_1$ ).

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### Proposition (GF - Pingali - Yao '20)

Let  $D = \sum_i n_i p_i$  be the effective divisor on  $\mathbb{P}^1$  corresponding to  $(L, \phi)$ . Assume that  $D$  is polystable with respect to the  $SL(2, \mathbb{C})$ -action.

Then, the existence of gravitating vortices is an open condition for  $\alpha > 0$ .

**Proof:** when  $D$  is stable, the proof follows from the moment map framework applying the Implicit Function Theorem.

For  $D = \frac{N}{2}p_0 + \frac{N}{2}p_1$  we apply the following theorem (cf. Lebrun-Simanca)

Theorem (Álvarez-Cónsul - GF - García-Prada '13, GF - Pingali - Yao '20)

Given a solution of the gravitating vortex equations with  $\alpha > 0$ , for any nearby  $\alpha' \sim \alpha$  there exists an *extremal pair* with coupling constant  $\alpha'$ .

The proof follows from the vanishing of the Futaki invariant ( $D$  polystable), which implies that an *extremal pair* with coupling constant  $\alpha'$  is a gravitating vortex. □



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- Need to rule out the formation of singularities on a sequence of solutions when taking  $\alpha_k \rightarrow \alpha_0 > 0$ .

This is achieved in several steps. The first is the following:

### Lemma (GF - Pingali - Yao '20)

A tuple  $(\Sigma, L, \phi, g, h, \alpha, \tau)$  solving the gravitating vortex equations with  $g(\Sigma) = 0$  is equivalent to  $(g, \eta, \Phi)$  such that

- 1  $g$  is a smooth Riemannian metric on  $S^2$ ,
- 2  $\eta$  is a smooth closed real 2-form on  $S^2$  such that  $\int_{S^2} \eta = 2\pi N$ ,
- 3  $\Phi \in C^\infty(S^2)$  is a non-negative function  $\Phi \geq 0$  (*state function*), vanishing precisely at the  $p_j \in S^2$ , and such that  $\log \Phi \in L^1_{loc}(S^2)$ ,

solving the RGV system

$$\eta + \frac{1}{2}(\Phi - \tau) \text{vol}_g = 0, \quad \Delta_g \log \Phi = (\tau - \Phi) - 4\pi \sum_j n_j \delta_{p_j} \quad (6)$$

$$S_g + \alpha(\Delta_g + \tau)(\Phi - \tau) = c.$$

$\Sigma = (S^2, J)$  is the Riemann surface determined by  $g$ ,  $|\phi|_h^2 = \Phi$ , and  $\eta = iF_h$ .

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The second step is the following key *a priori estimates*:

### Theorem (GF - Pingali - Yao '19)

Let  $(g, \eta, \Phi)$  be a smooth solution of RGV. Then,

①  $0 \leq \Phi \leq \tau$

②  $\frac{1}{2\pi} \int_{S^2} \Phi \text{vol}_g = \tau - 2N$

③ (scalar curvature estimate)

$$c \leq S_g \leq \frac{(3 + 2\alpha\tau)\tau}{2}$$

④ (state function estimate)

$$-\frac{\tau^2}{4} \leq -\Delta_g \Phi = \frac{|\nabla \Phi|^2}{\Phi} - \Phi(\tau - \Phi) \leq \frac{1}{\alpha} \left( \frac{3\tau}{2} - c \right)$$

The third step is given by *Cheeger-Gromov's Theory*:

### Proposition (GF - Pingali - Yao '19)

Let  $(g_n, \eta_n, \Phi_n)$  be a sequence of solutions of RGV with  $\alpha_n \rightarrow \alpha_0$ , where  $0 < \alpha_0 < 1/\tau N$ .

Then, there exists a sequence  $\sigma_n \in \text{SL}(2, \mathbb{C})$  (for a suitable complex structure) such that  $\sigma_n^*(g_n, \eta_n, \Phi_n)$  converges in  $C^{1,\beta}$  sense to a smooth solution of RGV with constant  $\alpha_0$  and divisor  $D_\infty \in \text{SL}(2, \mathbb{C}) \cdot D$ .

**Proof:** By a priori estimates, the volume is bounded along the sequence

$$2\pi \leq \text{Vol}_{g_n} \leq 2\pi e^{2\alpha_n \tau},$$

and the diameter of  $g_n$  is bounded from above by Bonnet's estimate

$$\text{diam}(S^2, g_n) \leq \frac{\pi}{\sqrt{c_n e^{-2\alpha_n \tau}}} =: D_n.$$

By Relative Volume Comparison, the volume ratio is bounded from below:

$$\frac{\text{Vol}_{g_n} B(p, r)}{\pi r^2} \geq \frac{\text{Vol}_{g_n} B(p, D_n)}{\pi D_n^2} \geq \frac{2}{D_n^2}, \quad \forall p \in S^2, r \in (0, D_n],$$

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By Cheeger-Gromov-Taylor we have an estimate on the injectivity radius:

$$\text{inj}(S^2, g_n) \geq i_0 > 0$$

independent of  $n$ .

Finally, by a priori estimates we have uniform bounds for  $S_n$  and  $|\nabla_n S_n|_n$  along the sequence. The statement follows by Cheeger-Gromov compactness combined with a slice theorem for complex structures.



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**¡Merci Beaucoup!**