Some geometric applications of Hodge theory

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I. Introduction

- Modern Hodge theory is both a subject of study in its own right and a subject that is used in many areas of current mathematical research, especially in but no means restricted to algebraic geometry. This talk will be an informal and partial overview of some of its uses with emphasis on those in algebraic geometry. We will also discuss some of the historical development of the subject; how did it originate and how did it get to its current state? Here the emphasis will be on the period up until the time of Hodge and will only touch on a few of the major milestones.

  In the appendix we have given a proof of the local Torelli theorem for I-surfaces (the "first" non-classical general type surface). The point is to illustrate one computational technique in Hodge theory.

Hodge theory relates to many areas in mathematics

- analysis
  - classical complex analysis
  - modern complex analysis (the $\overline{\partial}$-operator)
  - ODE's (Picard-Fuchs equations)
  - non-linear PDE's (non-abelian Hodge theory and special metrics on algebraic varieties)

- differential geometry — provides a tool for the uses of Hodge theory in algebraic geometry including vanishing theorems, general type and log general type properties of algebraic varieties including moduli spaces, hyperbolicity of particular varieties

- topology
  - smooth varieties
  - {general algebraic varieties, singular and/or non-complete}
  - intersection cohomology
  - {families of algebraic varieties; here all of the above enter the picture}

- Lie theory — {homogeneous complex manifolds, both compact and open; period domains and their compact duals (think of $\mathcal{H} \subset \mathbb{P}^1$); locally symmetric spaces and their generalizations; representation theory} and finally of course

- algebraic geometry; today we will only have time to discuss a very few aspects; will not be able to talk about the central problem in the subject (Hodge conjecture), mirror symmetry and other topics related to physics, character varieties, arithmetic algebraic geometry, ...

- Specifically, the use of Hodge theory to study moduli requires using geometric constructions arising from Hodge theory, frequently some type of Torelli property, either for the variety itself or for the singular ones that appear on the boundary of moduli spaces. A Hodge structure and some of its generalizations are given by linear algebra data. In some cases from the linear algebra data one may construct a geometric object from which properties of the variety may be determined. Although very classical this original use of Hodge theory (elliptic functions and Riemann's theta function) continues to find applications to geometric questions including moduli.

II. Origins of Hodge theory

Although there is no single work that one can say "here is the beginning of Hodge theory," an important part of its origins may be traced to the study of integrals

$$\int r(x, y(x))\, dx \tag{II.1}$$

of algebraic functions. Here $y(x)$ is the "function" given by an equation

$$f(x, y(x)) = 0$$

where $f(x, y)$ is an irreducible polynomial and $r(x, y) = p(x, y)/q(x, y)$ is a rational function. Such integrals arise in geometry, e.g. as arclength $\int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + (y'(x)^2))} dx$, or in mechanics where $\dot{y}(t)^2 = f(y(t))$ so that $y(t) = \int \sqrt{f(y(x))} dx$.

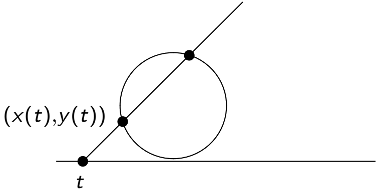
For illustrative purposes we will take

$$f(x,y) = y^2 - p(x) \tag{II.2}$$

where $p(x) = \prod_{i=1}^{2g+2}(x - a_i)$ is a polynomial with distinct roots. The integral (II.1) is understood to take place in the complex plane along a path γ avoiding the a_i and along which we have chosen a branch of $y(x) = \sqrt{p(x)}$. If $\deg f(x,y) = 2$, then such integrals can be evaluated in terms of "elementary functions," e.g., for $f(x,y) = x^2 + y^2 - 1$ then

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dx}{y} = \text{arcsin}. \tag{II.3}$$

The reason for $\deg f = 2$ is that $f(x, y) = 0$ is then a conic and can be parametrized by a line



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substituting gives for the integral

$$\int r(t)dt$$

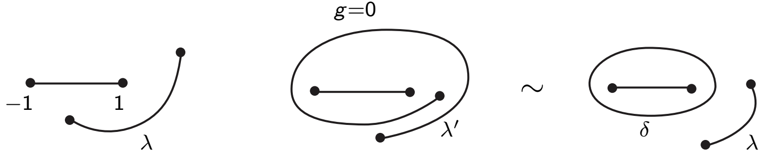
where $r(t)$ is a rational function and partial fractions may be used to evaluate it.

As soon as $\deg f \geqq 3$ the integral (II.1) can no longer be understood in terms of elementary functions. For example

$$\text{(II.4)} \qquad \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \qquad k \neq \pm 1$$

turns up in the arc length of an ellipse.

Before continuing we want to make an observation that has both topological and analytic meaning.
For $f(x, y) = y^2 - p(x)$ as in (II.2) and $\omega = dx/y$ we have



$$\int_{\lambda'} \omega = \int_{\lambda} \omega + \int_{\delta} \omega = \int_{\lambda} \omega + c$$

where $c = \int \omega$ is a period of the integral.

If we invert the integral by setting

$$u = \int_{(x_0,y_0)}^{(x(u),y(u))} \omega \tag{II.5}$$

then

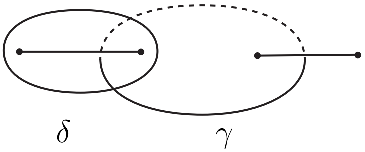
$$x(u+c) = x(u), \quad y(u+c) = y(u).$$

In fact, in (II.3) we have $c = 2\pi$ and this shows that defining $x(y), y(u)$ by (II.5) parametrizes the circle by arclength and gives the periodicity of $(\sin u, \cos u)$. Note that taking the derivative of (II.5) gives

$$x'(u) = y(u).$$

If we try to do the same thing with (II.4), then we arrive at a picture

$g = 1$



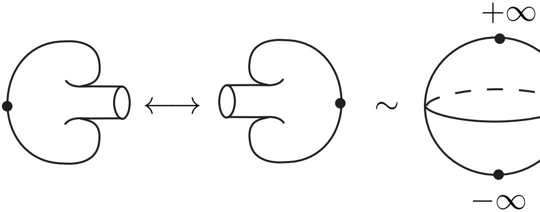
where inversion of the integral leads to doubly periodic functions $x(u), y(u)$.

Analytically

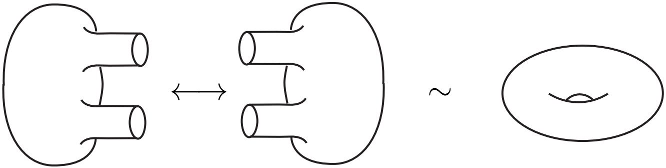
- for $g = 0$ the integral $\int_\gamma \omega \to \pm\infty$ as $\partial\gamma \to \infty$ (in fact, $\omega \sim \pm(dt/t^{1/2})$ for $x = 1/t$);
- for $g = 1$ the integral $\int_\gamma \omega < \infty$ for any γ (in fact, for $k \geqq g - 1$ we have $\int_\gamma \frac{x^k dx}{y} < \infty$)

Topologically

- for $g = 0$ we have



- for $g = 1$ we have



What about a general integral (II.1)? Let $h(x, y, t)$ depend rationally on t (e.g., $h(x, y, t) = ax + by + ct$), and

$$(f(x, y) = 0) \cap (h(x, y, t) = 0) = \sum_i (x_i(t), y_i(t)) = \sum_i p_i(t).$$

The decisive result was given by

Abel ([A]): $\frac{d}{dt}\left(\sum_i \int_{p_0}^{p_i(t)} \omega\right) = r(t)$ *is a rational function of* t.

Thus the *abelian sum* $\sum_i \int_{p_0}^{p_i(t)} \omega$ is an elementary function. If we define $x(u), y(u)$ by inversion

$$u = \int_{(x_0, y_0)}^{(x(u), y(u))} \omega,$$

then it turns out that one can express $x(u + c), y(u + c)$ in terms of $x(u), y(u)$. Moreover, for $f(x, y) = y^2 - p(x)$ we have $x'(u) = y(u)$. For $g = 0, 1$ we have the periodicity or double periodicity of $x(u), y(u)$.

If $r(t)dt$ has no residues so that $\int r(t)dt$ is a rational function, then $x_i(u+c) = R_i(x_1(u), x_1'(u), \ldots, x_d(u), x_d'(w))$. Taking $h(x, y, t)$ to be linear as above one may obtain the addition theorem

$$x(u+\tilde{u}) = R(x(u), x'(u), x(\tilde{u}), x'(\tilde{u})).$$

- Abel ([A]) defined the *genus* of $X := \{f(x, y) = 0\}$ as the dimension of the space $H^0(\Omega_X^1)$ of ω's with $\int \omega < \infty$. Then Riemann ([R]) proved
  - $g = \left(\frac{1}{2}\right) b_1(X) = \frac{1}{2} (\dim H_1(X, \mathbb{Z}))$
  - $H^1(X, \mathbb{C}) \cong H^0(\Omega_X^1) \oplus \overline{H^0(\Omega_X^1)}$.

  His argument for the second used $\int \omega \wedge \omega = 0$ (because $dx \wedge dx = 0$) and $i \int \omega \wedge \overline{\omega} > 0$ (because $i\, dx \wedge d\bar{x} > 0$). This was the beginning of Hodge theory.

- Finally a few words about Picard (cf. [PS]). He studied algebraic surfaces

$$X = \{f(x, y, z) = 0\}$$

by looking at the pencil of algebraic curves

$$Y_z = \{f(x, y, z) = 0, z = \text{constant}\}.$$

Then he proved

  - Y_z is connected ($H_0(Y_z, \mathbb{Z}) \xrightarrow{\sim} H_0(X, \mathbb{Z})$)
  - $H_1(Y_z, \mathbb{Z}) \twoheadrightarrow H_1(X, \mathbb{Z})$.

On X he considered differentials

  - $\varphi = p(x, y, z)dx + q(x, y, z)dy, \quad \int \varphi < \infty$
  - $\omega = r(x, y, z)dx \wedge dy, \quad \iint \omega < \infty$.

For the first Picard showed that $d\varphi = 0$ and from this inferred

  - $H^1(Y, \mathbb{C}) \cong H^0(\Omega^1_X) \oplus \overline{H^0(\Omega^1_X)}$

This is because if $d\varphi \neq 0$, then

$$0 \neq \int d\varphi \wedge \overline{d\varphi} = \int d(\varphi \wedge \overline{d\varphi}) = 0$$

and similarly we cannot have $\varphi = d\eta$. For 2-forms he defined what we now call the *Picard number* ρ and showed that

(II.6) ▶ $H^2(X, \mathbb{C}) \supset H^0(\Omega_X^2) \oplus \overline{H^0(\Omega_X^2)} \oplus (\mathbb{C}^\rho)$.

For the third term here the Lefschetz (1,1) theorem ([L]) gives

$$H_2(X, \mathbb{Z}) \cap H^0(\Omega_X^2)^\perp = NS(X).$$

The full identification of the term to make $\supset$ into $=$ using $H^{1,1}(X)$ had to await the work of Hodge ([H]).

III. Objects of Hodge theory (cf. [CM-SP], [V1], [V2])

- *Polarized Hodge structure* (PHS): (V, Q, F)
  - $V = \mathbb{Q}$-vector space
  - $Q : V \otimes V \to \mathbb{Q}$
  - $F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$

  where

  $$F^p \cap \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}, \quad 0 \leqq p \leqq n.$$

  For $V^{p,q} = F^p \cap \overline{F}^q$ we have the Hodge decomposition

  - $V_{\mathbb{C}} = \overset{p+q=n}{\oplus} V^{p,q}, \quad \overline{V}^{p,q} = V^{q,p}$
  - $F^p = \overset{p' \geqq p}{\oplus} V^{p',q}$

This defines a Hodge structure of weight n. The polarization arises from the first and second Hodge-Riemann bilinear relations, which are generalizations of the ones given above for algebraic curves and which we don't need to make explicit here. Most of the deeper results in Hodge theory require the existence of a polarization.

Example: For X a smooth algebraic variety, $H^n(X, \mathbb{Q})$ has a Hodge structure of weight n. If $X \subset \mathbb{P}^N$, so that $L = \mathcal{O}_X(1)$ is an ample line bundle, there is a polarization on $H^n(X, \mathbb{Q})$.

- *Mixed Hodge structure* (MHS): (V, W, F)
  - $V = \mathbb{Q}$-vector space
  - $W_0 \subset \cdots \subset W_m = V$
  - $F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$

  where F induces a weight k Hodge structure on $\mathrm{Gr}_k^W(V) = W_k / W_{k-i} := H^k$.

Although $V_{\mathbb{C}}$ does not have a Hodge decomposition, there is the canonical Deligne decomposition

$$V_{\mathbb{C}} = \overset{p+q\leqq m}{\oplus} I^{p,q}$$

where

$$\begin{cases} W_{k,\mathbb{C}} = \overset{p+q\leqq k}{\oplus} I^{p,q} \\ \overline{I}^{p,q} \equiv I^{q,p} \bmod W_{p+q-2,\mathbb{C}} \end{cases}$$

which gives that

$$H^k \cong \overset{p+q=k}{\oplus} I^{p,q}.$$

There is a definition of a graded polarized MHS. Again this property is frequently present in applications of mixed Hodge theory.

In practice we will have a lattice $V_{\mathbb{Z}} \subset V$ with induced lattices on the H^k's. Thus a MHS is a successive extension of Hodge structures with the first level being a direct sum of terms

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H^k, H^{k-1}) = \frac{\mathrm{Hom}(H^k, H^{k-1})}{F^0 \,\mathrm{Hom}(H^k, H^{k-1}) + \mathrm{Hom}_{\mathbb{Z}}(H^k, H^{k-1})}.$$

In general the set $\mathcal{E}_k$ of at most k-fold iterated extensions of HS's with fixed H^k's gives a sequence of fibrations $\mathcal{E}_k \to \mathcal{E}_{k-1}$ with typical fibre the direct sum of

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H^\ell, H^{\ell-k})\text{'s.}$$

- *Variation of Hodge structure* (VHS): $(\mathbb{V}, \mathcal{F}, \nabla; B)$
  - $\mathbb{V} = \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{Q}$ is a local system over a quasi-projective variety B
  - $\mathcal{F}$ is a filtration $\mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \cdots \subset \mathcal{F}^0 = \mathcal{V} = \mathbb{V} \otimes \mathcal{O}_B$ by sub-bundles that induces a Hodge structure on each fibre
  - $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_B$ satisfies $\nabla \mathbb{F} = 0$ and the infinitesimal period relation (IPR)

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_B.$$

It is understood that there is a horizontal bilinear form $Q : \mathbb{V} \otimes \mathbb{V} \to \mathbb{Q}$ that polarizes the Hodge structure on each fibre.

Example: $\mathcal{X} \xrightarrow{\pi} B$ is a smooth family of projective algebraic varieties X_b and

  - $\mathbb{V} = R^n_\pi \mathbb{Q}_{\mathcal{X}}$
  - $\mathcal{F}_b$ gives the Hodge structure on $H^n(X_b, \mathbb{C})$
  - ∇ is the Gauss-Manin connection; $\nabla^2 = 0$.

- *Infinitesimal variation of Hodge structure (IVHS)*: (E, T, θ)
  - $E = \oplus E^p$
  - $\theta : E^p \to E^{p-1} \otimes T^*$
  - $\theta \wedge \theta = 0$.

  Thus E is a $\mathrm{Sym}\, T$-module.

Example: $E^p = \mathrm{Gr}^p_{F_b}(\mathcal{V}_b)$, $T = T_b B$ and θ is induced by ∇. It is essentially given by the differential of the period mapping (which we won't define here).

- *Limiting mixed Hodge structure (LMHS):* $(V, W(N), F)$
  - $N \in \mathrm{End}(V)$ satisfying $N^{m+1} = 0$ gives a unique weight filtration $W_k(N)$ satisfying

$$\begin{aligned} N :& W_k(N) \to W_{k-2}(N) \\ N^k :& \mathrm{Gr}^{W(N)}_{m+k}(V) \cong \mathrm{Gr}^{W(N)}_{m-k}(V) \\ N :& F^p \to F^{p-1}. \end{aligned}$$

Again we assume there exists a $Q : V \otimes V \to \mathbb{Q}$ and $N \in \mathrm{End}_Q(V)$. Then there are induced bilinear forms

$$Q_k : \mathrm{Gr}^{W(N)}_k(V) \otimes \mathrm{Gr}^{W(N)}_k(V) \to \mathbb{Q}$$

and these are assumed to polarize the Hodge structures on the $\mathrm{Gr}^{W(N)}_k(V)$.

Example: For $\Delta^* = \{0 < |t| < 1\}$ suppose we have a VHS $(\mathbb{V}, \mathcal{F}, \nabla; \Delta^*)$ over the punctured disc. For $V = \mathbb{V}_{t_0}$ it is known that the monodromy

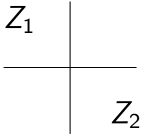
$$T : V \to V$$

is quasi-unipotent; i.e., $T = T_s T_u$ where $T_s^\ell = I$ and $T_u = e^N$ with $N^{m+1} = 0$ for some $m \leqq n$. Then Schmid proved that there is a LMHS $H^n_{\lim}$ (actually an equivalence class of such parametrized by $T^*_{\{0\}}\Delta$).

This example has been extended by Cattani-Kaplan-Schmid to the case when the parameter space is $\Delta^{*a} \times \Delta^b := B$. For our purposes it will be convenient to assume the local monodromies around the generators of $\pi_1(\Delta^{*a})$ are unipotent. In this case there is a canonical Deligne extension of the Hodge bundles to $\mathcal{F}^p_e \to \Delta^a \times \Delta^b := \overline{B}$ where the Gauss-Manin connection satisfies

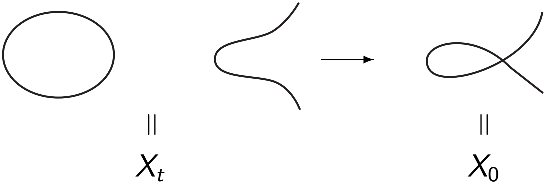
$$\nabla : \mathcal{F}^p_e \to \mathcal{F}^{p-1}_e \otimes \Omega^1_{\overline{B}}(\log Z)$$

where $Z \subset \overline{B}$ is the normal crossing divisor with one of the coordinates in the Δ^a being set equal to 0:



$$Z = Z_1 + Z_2$$

Example (continued): $\mathcal{X} \to \Delta$ is a projective family of varieties that is smooth over Δ^*



Then we have $\lim_{t \to 0} H^n(X_t) := H^n_{\lim}$. The relation of $H^*_{\lim}$ to $H^n(X_0)$ is given by the Clemens-Schmid long exact sequence where X_0 is a NCD. When X_0 has slc singularities there is a variant of Clemens-Schmid discussed in [K].

Note that the left-hand side is independent of the particular singular fibre X_0 over the origin.

- In the global situation where one has a pair $(\overline{B}, Z)$ consisting of a smooth projective variety $\overline{B}$ with $Z \subset \overline{B}$ a reduced normal crossing divisor, given a VHS over B with unipotent monodromies around the irreducible branches of Z the above local discussion applies to give a canonical extension of the VHS to $\overline{B}$.*

*Again the assumption of unipotent monodromies is convenient for expository purposes; in interesting geometric situations it frequently does not occur (e.g., algebraic surfaces acquiring normal (and hence isolated) singularities).

IV. Some geometric applications of Hodge theory

- In some situations one may associate to a PHS, an IVHS or a LMHS a geometric construction, and when the Hodge theoretic object arises from geometry this construction may help understand the geometry.

The classical example: Associated to a weight $n = 1$ PHS (V, Q, F) is

- a compact complex torus

$$J = (F^1 \backslash V_{\mathbb{C}})/V_{\mathbb{Z}};$$

- a line bundle $L \to J$ with $c_1(L) = Q \in (\Lambda^2 V_{\mathbb{Z}})^* \cong H^2(J, \mathbb{Z})$;
- it follows from the Hodge-Riemann bilinear relations that L is holomorphic and positive in the differential geometric sense, hence it is ample. If Q is unimodular, then $h^0(J, L) = 1$ and there is a non-zero section θ giving a divisor $\Theta \subset J$.

This is the case when the PHS arises from the H^1 of a smooth algebraic curve X. There is an Abel-Jacobi map

$$u : X \to J(X)$$

given for $p \in X$ by the linear function on $H^0(\Omega^1_X) \cong (F^1 \backslash V_{\mathbb{C}})^*$

$$\omega \to \int_{p_0}^{p} \omega \text{ modulo periods.}$$

Using Abelian sums as were encountered in Abel's theorem this mapping extends to

$$\begin{array}{ccc} X^{(d)} & \longrightarrow & J(X) \\ \cup & & \cup \\ \sum_i p_i & \longrightarrow & \sum_i u(p_i) = \sum_i \int_{p_0}^{p_i} \omega \end{array} .$$

Riemann ([R]) proved that, up to a translation,

$$u(X^{(g-1)}) = \Theta.$$

From this one may determine much of the geometry of X, culminating in Torelli's theorem that the PHS on $H^1(X)$ determines X.

In general it is not possible to associate a geometric object to a PHS. The reason is the differential constraint imposed on a VHS by the IPR. Only when the period domain is Hermitian symmetric, in which case the IPR is trivial and the PHS is a (Tate twist of) one of weight $n = 1$ or of weight $n = 2$ with Hodge number $h^{2,0} = 1$, can we construct naturally an algebro-geometric object from just a PHS. In the higher weight case one frequently may use an IVHS as a surrogate for the Θ-divisor. A recent case of this is given by the

Example (Shepherd-Barron):[†] If $X \to C$ is an elliptic surface with no multiple fibres and with

$$h^{2,0} \geqq h^{1,0} + 3,$$

then the PHS on $H^2(X)$ generically determines X.
The IVHS used here is the first variation of the PHS on $H^2(X)$, which in a very interesting way is interpreted via the ramification of the j-function map $C \to \overline{\mathcal{M}}_1$.

A classical example of the use of IVHS is the theorem of Donagi-Green ([GMV]) concerning smooth hypersurfaces

$$\{F(x_0, x_1, \ldots, x_{n+1}) = 0\} = X \subset \mathbb{P}^{n+1}.$$

[†] (arXiv:2009.03683v, 9Sept2020)

In this case, for $n \geqq 2$ the IVHS is given by a homogeneous subring $R \subset \mathbb{C}(x_0, \ldots, x_{n+1})/J_F$ where $J_F = \{F_{x_0}, \ldots, F_{x_{n+1}}\}$ is the Jacobian ideal of F. Except for a few cases of $\deg F$, it is shown that R determines J_F up to the action of PGL_{n+1}.

An even more classical example arises in the case of algebraic curves X of genus $g \geqq 2$. In this case the IVHS turns out to be equivalent to the map

$$\mathrm{Sym}^2 H^0(K_X) \to H^0(2K_X).$$

For $g \geqq 3$ and X non-hyperelliptic this mapping is surjective (Noether) (which then implies local Torelli) with kernel the space $I_2(X)$ of quadrics through the canonical curve $X \subset \mathbb{P}^{g-1}$. For $g \geqq 5$, again by Noether

$$\bigcap_{Q \in I_2(X)} Q = X$$

and from this one infers generic global Torelli for $g \geqq 5$.

In these examples involving full use of the two Hodge-Riemann bilinear relations is not made. In the following both the IVHS and the bilinear relations are used.

Example: A recent example that is in some ways reminiscent of the Θ-divisor arises from a pair $(\overline{B}, Z)$ together with a VHS over $B = \overline{B} \backslash Z$ that has been canonically extended to $\overline{B}$. We have noted that at each point of Z the level 1 extension data $\mathcal{E}_1$ is a direct sum of compact, complex tori each of whose tangent spaces is a Hodge structure of weight-1 with Hodge decomposition

$$\text{(III.1)} \quad (k-1,-k)\oplus\cdots\oplus\underbrace{(0,-1)\oplus(-1,0)}\oplus\cdots\oplus(-k,k-2).$$

The level 2 extension data fibres over $\mathcal{E}_1$ with fibre a direct sum of $\mathrm{Ext}^1_{\mathrm{MHS}}(H^k, H^{k-2})$'s. Here the tangent space is a Hodge structure of weight -2 with Hodge decomposition

$$(k-2,-k)\oplus\cdots\oplus\underbrace{(-1,-1)}\oplus\cdots\oplus(-k,k-2).$$

Using $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-1),\mathbb{Q})\cong\mathbb{C}^*$ one sees that the Hodge part over the brackets gives line bundles $L_M\to E$ whose curvature is positive on the part of $T\mathcal{E}_1$ over the bracket in (III.1). As we vary along a subvariety $A\subset Z$ along which the associated graded $\oplus H^k$ to the limiting mixed Hodge structures remains constant, using the IPR we obtain an ample line bundle $L_M\to A$. A central geometric result is that (cf. [GGR])

$$\boxed{L_M\big|_A\cong\bigoplus_i\langle M,N_i\rangle\, N^*_{Z_i/\overline{B}}\big|_A.}$$

This gives the picture of the VHS at infinity

$$\begin{array}{ccc}
\left\{\begin{matrix}\text{subvarieties of}\\ \text{abelian varieties}\end{matrix}\right\} & \longrightarrow & (\mathcal{V}_e, \mathcal{F}_e) \\
 & & \Big\downarrow \left\{\begin{matrix}\text{associated graded}\\ \text{to the LMHS}\end{matrix}\right\} \\
 & & \left\{\begin{matrix}\text{variations of}\\ \text{PHS's}\end{matrix}\right\}
\end{array}$$

where using the above result in the box the co-normal bundle to the fibres in $\overline{B}$ is expressed by Θ-line bundles along the fibres. We believe that the above picture serves as **the** governing property in the global understanding of the Hodge structure on the boundary of the canonically extended VHS ([GGR]).

Geometric example: Related to the preceding, let $\mathcal{M}$ be a KSBA moduli space of surfaces of general type and with canonical completion $\overline{\mathcal{M}}$ where the boundary $\partial\mathcal{M} = \overline{\mathcal{M}} \backslash \mathcal{M}$ parametrizes surfaces X with semi-log-canonical singularities (cf. [K]). A part $\mathcal{N}_e$ of the boundary corresponds to normal Gorenstein X's with simple elliptic singularities. In contrast to the case of algebraic curves where $\overline{\mathcal{M}}_g$ is essentially smooth, $\mathcal{N}_e \subset \partial\mathcal{M}$ is generally highly singular and consideration of the extension data in the LMHS suggests a natural partial desingularization of $\overline{\mathcal{M}}$ along $\mathcal{N}_e$. This is explained in the notes [G2], where it is also illustrated that this phenomenon may also extend to non-Gorenstein isolated singularities and to non-normal X's as well.

This is the first non-classical example I am aware of where an actual geometric object may be constructed from a (generalized) Hodge structure alone, in this case a LMHS. It is an IVHS because from the work of Cattani-Kaplan-Schmid a LMHS may be smoothed to a 1-parameter of ordinary PHS's.

Geometric example (continued): As above suppose we have a KSBA moduli space $\mathcal{M}$ for general type surfaces and with canonical completion $\overline{\mathcal{M}}$. Assume for simplicity that $\mathcal{M}$ is smooth and that a general point of $\mathcal{M}$ corresponds to a smooth regular surface. One may ask

- can Hodge theory suggest what surfaces X appear on the boundary $\partial\mathcal{M} = \overline{\mathcal{M}}\backslash\mathcal{M}$?
- Can Hodge theory suggest how one might construct a desingularization $\widetilde{\overline{\mathcal{M}}}$ of $\overline{\mathcal{M}}$?

As discussed and illustrated in [G2] the answer to both questions is positive. For the first, a general limiting mixed Hodge structure has

$$N^2 = 0, \quad \operatorname{rank} N = 2.$$

Thus the LMHS has associated graded $(H^1, H^2, H^1(-1))$ where $H^1 = H^1(C)$ for a smooth elliptic curve C. Given a KSBA degeneration $\mathcal{X} \to \Delta$ where X_t is smooth for $t \neq 0$ and X_0 is a normal surface corresponding to a point of $\partial\mathcal{M}$, what is suggested is that $X_0 = (X, p)$ where p is a simple elliptic singularity of a surface X and where the resolution of that singularity is $(\widetilde{X}, C) \to (X, p)$ where $\widetilde{X}$ is smooth and $C \subset \widetilde{X}$ is an elliptic curve.

For the desingularization of $\overline{\mathcal{M}}$ one needs to do a semi-stable reduction

$$\widetilde{\mathcal{X}} \to \widetilde{\Delta}$$

of the family $\mathcal{X} \to \Delta$. Since $N^2 = 0$, Clemens-Schmid suggests that the central fibre $\widetilde{X}_0$ should have a double curve C and no triple points. The simplest possibility is that

$$\widetilde{X}_0 = \widetilde{X} \bigcup_C Y$$

where $\widetilde{X}$ is as above and Y is a smooth surface containing C. Since C is a smooth elliptic curve we might try a smooth cubic $C \subset \mathbb{P}^2$. The normal bundle $N_{C/\widetilde{X}}$ has degree $d = -C^2$ while $N_{C/\mathbb{P}^2} \cong \mathcal{O}_C(3)$. To achieve the necessary condition

$$N_{C/\widetilde{X}} \cong \check{N}_{C/Y}$$

for smoothability, we must blow up $9 - d$ points p_i on C. Since for a smoothable elliptic singularity we have

$$1 \leqq d \leqq 9$$

Y is a del Pezzo surface. Moreover from $\widetilde{X}_0$ as above we can construct the potential limiting mixed Hodge structure, and a standard computation gives that

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(-1), H^2)$$

contains a factor constructed from the subspace $\mathrm{Hg}^1(Y, \mathbb{Z})$ in H^2. It then follows that the information contained in the level 1 extension data in the LMHS is essentially the

$$\mathrm{AJ}_C(p_i - p_j) \in J(C).$$

This tells us which points p_i on $C \subset \mathbb{P}^2$ to blow up to construct Y.

Of course the above is heuristic, but hopefully it does the tight interplay between Hodge theory and geometry and illustrates how a geometric construction from a IVHS may be used to study moduli.

Topology: The above examples of uses of Hodge theory were to geometric constructions arising from linear algebra Hodge theoretic data. The original application of Hodge theory, meaning now the existence of a functorial Hodge structure on the cohomology of a smooth projective variety $X^n \subset \mathbb{P}^N$, were to topology. Letting $Y = \mathbb{P}^{N-1} \cap X$ be a general hyperplane section there is the

(III.2) Lefschetz theorem ([L]): *The induced mapping*

$$H^q(X,\mathbb{Z}) \to H^q(Y,\mathbb{Z})$$

is an isomorphism for $q \leqq n-2$ and is injective for $q = n-1$.

This result was proved by Picard ([PS]) when $n = 2$ (algebraic surfaces), and Lefschetz's argument for the general case was an extension of that of Picard. It made use of Lefschetz pencils (Y_t) given by the $\mathbb{P}^1$ of hyperplane sections containing a general $\mathbb{P}^{N-2}$. There are finitely many singular Y_{t_α}'s where the hyperplane is tangent to X, and the monodromy around these (Picard-Lefschetz transformations) plays a central role in the analysis. The result is a topological one; no use is made of Hodge theory.

The other result of Lefschetz concerns the mapping

$$h^k : H^{n-k}(X, Q) \to H^{n+k}(X, Q) \tag{III.3}$$

where $h \in H^2(X, Q)$ is dual to the homology class defined by Y.

(III.4) **Hard Lefschetz theorem ([L] and [H]):**
The mapping h^k in (III.3) *is an isomorphism.*

The reason for the “hard” is that in many senses this result is deeper than (III.2). In fact, Lefschetz’s original topological argument was incomplete. It was partly in seeking to give a proof of (III.4) that Hodge [H] originated Hodge theory; the result itself is a Hodge-theoretic, not a topological, one. Interestingly Picard stated and gave a correct proof of (III.4) when $n = 2$. His argument used the Poincaré complete reductibility theorem ([P]) for abelian varieties, applied here to the family of Jacobian varieties $J(Y_t)$. The complete reducibility theorem is basically a Hodge-theoreic result and its higher dimensional analogue had to await the construction of PHS's on the $H^{n-1}(Y_t)$ for $n \geqq 3$.

Vanishing theorems

The Italian algebraic geometers had extraordinary geometric intuition and deep knowledge of examples. In its more mature period there were things they intuitively suspected but were not able to fully prove, and even some mistakes appeared. It has been said that what they were missing was H^1. The meaning of this is illustrated by the

Example: Let $L \to X$ be a line bundle over a smooth algebraic surface and $C \subset X$ a curve. Various forms of the issue of showing that all of $H^0(C, L)$ comes by restriction from $H^0(X, L)$ kept arising. The Italians knew that this is not true in general. With the advent of sheaf theory and the long exact cohomology sequence we now know this will be OK if one has the vanishing theorem $H^1(X, L(-C)) = 0$.[‡]

[‡]This issue does not arise as much for points on a curve because in the case the dual of H^1 is an H^0.

Thus one needs (a) the formalism of sheaf cohomology, and (b) vanishing theorems. The thesis of this last part of the talk is

> *The Lefschetz theorem* (III.2) *plus the existence of Hodge structures on ordinary cohomology imply vanishing theorems.*

More precisely, one needs the existence of the Hodge structure

$$H^r(X, \mathbb{C}) \cong \overset{p+q=r}{\oplus} H^{p,q}(X) \tag{III.5}$$

plus the isomorphism

$$H^{p,q}(X) \cong H^q(\Omega_X^p). \tag{III.6}$$

This principle is well known since the 1950's and over the years has attained a much extended and more widely applicable version (cf. [SS], [EV] and the lecture notes [G2]). Here we will simply illustrate it in a very special case.

Let $X \subset \mathbb{P}^N$ and $Y = P^{N-1} \cap X$ be as above and $L = \mathcal{O}_X(1)$ the restriction to X of the standard hyperplane bundle. Then the Kodaira vanishing theorem is

$$\text{(III.7)} \qquad H^q(X, \check{L}) = 0 \;\text{ for }\; q < n = \dim X.$$

Proof: If $s \in H^0(X, L)$ is the restriction to X of the linear form that defines $Y \subset X$, we have

$$\text{(III.8)} \qquad 0 \to \check{L} \xrightarrow{s} \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

Then by (III.5) and (III.6)

$$\begin{array}{ccccc} H^q(\mathcal{O}_X) & \cong & H^{0,q}(X) & \subset & H^q(X, \mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \\ H^q(\mathcal{O}_Y) & \cong & H^{0,q}(Y) & \subset & H^q(Y, \mathbb{C}). \end{array}$$

The right-hand arrow is injective for $q \leqq n-1$ and then (III.7) follows from the long exact cohomology sequence of (III.8).

This talk began with some historical discussion and we would like to conclude with a final historical point. One of the main classical results established by Picard and the Italian algebraic geometers concerned a general hyperplane section $C \subset X$ of a smooth surface X. The adjunction formula is

$$K_X + C\big|_C \cong K_C$$

and the result is

The cokernel of the map $H^0(K_X + C) \to H^0(K_C)$ has dimension equal to the irregularity $q := h^0(\Omega^1_X)$ of X.

This follows from the exact cohomology sequence of

$$0 \to K_X \to K_X + C \to K_C \to 0$$

once one we know that

- $h^1(K_X + C) = 0$
- $h^1(K_X) = h^0(\Omega^1_X)$.

For $L = [C]$ as above, the first of these follows from the above vanishing theorem and Kodaira-Serre duality

$$H^1(K_X + C) \cong H^1(\check{L})^\vee.$$

The second follows from Hodge theory and Hard Lefschetz

$$H^0(\Omega^1_X) \overset{h}{\xrightarrow{\sim}} H^1(\Omega^2_X).$$

The dual of this map is part of the map $H_3(X) \overset{\check{h}}{\to} H_1(C) \to H_1(X)$ where the image of $\check{h}$ is the space of invariant cycles as C varies in a general pencil.

I think that once one has the Poincaré complete reducibility theorem this argument, which is given in [PS], may be basically correct.

Appendix: A Torelli theorem

An I-surface is a smooth general type surface X that satisfies $K_X^2 = 1$, $q(X) = 0$ and $p_g(X) = 2$. These surfaces are well known classically. We shall derive some of their properties, including the

Theorem

(i) *The local Torelli property is valid for any I-surface.* (ii) *The period mapping is*

$$\Phi : \mathcal{M}_I \to \Gamma \backslash D_I$$

where

$$\dim \mathcal{M}_I = \left(\frac{1}{2}\right)(\dim D_I - 1) = 28,$$

and where $\Phi(\mathcal{M}_I)$ is a maximal integral manifold of the IPR on D_I, which is a contact system.

The only other similar example we are aware of where the IPR may locally be explicitly "integrated" is a few Calabi Yau

(i) Projective realization of an I-surface

We denote by $Q_0 \subset \mathbb{P}^3$ the quadric $\{x_0 x_2 = x_1^2\}$ with singular point $p = [0, 0, 0, 1]$.

Proposition

A general I-surface X is realized via the bi-canonical map as a 2:1 covering of Q_0 branched over p and $V \cap Q_0$ where $V \subset \mathbb{P}^3$ is a general quintic surface not passing through p. Via its 5-canonical map it is realized as a hypersurface

$$z^2 = F_5(t_0, t_1, y)z + F_{10}(t_0, t_1, y)$$

in $\mathbb{P}(1, 1, 2, 5)$ with coordinates $[t_0, t_1, y, z]$ and where F_k is a weighted homogeneous polynomial of degree k.

Proof. The pencil $|K_X|$ has no fixed component so by Bertini a general $C \in |K_X|$ is a smooth curve of genus $g = \frac{1}{2}(K_X \cdot C + C^2) + 1 = 2$. We choose a basis t_0, t_1 for $H^0(K_X)$ such that $C = \{t_0 = 0\}$. From the general formula

$$h^0(mK_X) = \left(\frac{m(m-1)}{2}\right) K_X^2 + \chi(\mathcal{O}_X), \qquad m \geqq 2$$
$$= \frac{m(m-1)}{2} + 3$$

we have $h^0(2K_X) = 4$. Setting $K_C^{1/2} = K_X\big|_C$, from the exact cohomology sequence of

$$0 \to (m-1)K_X \xrightarrow{t_0} mK_X \to K_C^{m/2} \to 0$$

we may choose a basis $t_0^2, t_0 t_1, t_1^2, y$ for $H^0(2K_X)$ where the restrictions of t_1^2, y to C give a basis for $H^0(K_C)$. It follows that $(2K_X)$ is base point free and that using the above basis as homogeneous coordinates we have

$$\varphi_{2K_X} : X \to Q_0 \subset \mathbb{P}^3.$$

Since $t_0(p) = t_1(p) = 0$, it follows that $y(p) \neq 0$, so that near p

$$\varphi_{K_C} = t_1^2/y$$

vanishes to 2^{nd} order at p. Thus φ_{K_C} is a 2:1 mapping to one of the rulings of Q_0 which is branched at the vertex p and at 5 residual points on the ruling. It follows that

$$\varphi_{2K_X} : X \to Q_0$$

is a 2:1 map branched over $p + V$ when $V \in |Q_0(5)|$ does not pass through p.

For the second part of the theorem, using the above formula for the $h^0(mK_X)$ and the exact cohomology sequences arising from the above exact sheaf sequence we have

- $H^0(2K_X)$ has dimension 4 with basis given by the weighted degree 3 monomials in t_0, t_1, y where t_0, t_1 have weight 1 and y has weight 2;
- $H^0(4K_X)$ has basis the degree 4 weighted monomials in t_0, t_1, y;
- $H^0(5K_X)$ has basis the degree 5 weighted monomials in t_0, t_1, y plus one additional weight 5 generator z.

For the pluri-canonical ring $R_X = \overset{m}{\oplus} H^0(mK_X)$ we have

$$R_X \supset \mathbb{C}[t_0, t_1, y] \oplus z\mathbb{C}[t_0, t_1, y].$$

The two summands on the right are the $\pm$ 1 eigenspaces for the action of the involution $\tau : X \to X$ induced by the sheet interchange associated to the branched covering $\varphi_{2K_X} : X \to Q_0$. Computing dimensions we see that equality holds in the above inclusion, from which it follows that for R_X there is a generating relation

$$z^2 = F_5(t_0, t_1, y)z + F_{10}(t_0, t_1, y). \qquad \square$$

For later reference we note that since

$$X \cong \mathrm{Proj}\, R_X$$

it follows that the image $\varphi_{2K_X}(X) \subset \mathbb{P}(1, 1, 2, 5)$ is a smooth surface biregularity equivalent to X.

(ii) Alternate realization of an I-surface

We set

$$F = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2), \quad \xi = \mathcal{O}_{\mathbb{P}F}(1).$$

Then the linear system $|\xi|$ gives the desingularization map

$$f : \mathbb{P}F \to Q_0.$$

With standard notations, we have a unique up to scaling section $x \in |\xi - 2\mathfrak{h}|$ with divisor $S \cong \mathbb{P}^1$; then the self-intersection $S^2 = -2$ and f contracts this -2 curve to the node $p \in Q_0$.

Proposition

Denoting by $\hat{X}$ the blow up of X at the base point p of $|K_X|$, we have a mapping

$$g : \hat{X} \to \mathbb{P}F$$

which is a 2:1 covering branched over $\hat{B} = S + \hat{V}$ where $\hat{V} \in |5\xi|$.

Proof. We first note that if a section $\beta \in H^0(\mathbb{P}F, [\hat{B}])$ defines $\hat{B}$, where $[\hat{B}] = 6\xi - 2k\mathfrak{h}$, then for $L = 3\xi - k\mathfrak{h}$ with $L^2 = [\hat{B}]$ we may construct the embedding

$$\hat{X} \to \mathcal{O}_{\mathbb{P}F} \oplus L$$

where $\hat{X} = \{1 \oplus \lambda(q) : q \in \mathbb{P}F \text{ and } \lambda(q)^2 = \beta(q) \in L_q\}$. Using

$$\begin{cases} K_{\hat{X}} = \pi^*(K_{\mathbb{P}F} \otimes L) \\ K_{\mathbb{P}F} = -2\xi \text{ and } L = 3\xi - k\mathfrak{h} \\ h^0(K_{\hat{X})} = 2 \end{cases}$$

we find that $k = 1$ and $\hat{B} \in |6\xi - 2\mathfrak{h}|$. Using that p is a branch point of all $C \in |K_X|$, writing

$$\hat{B} = S + \hat{V}$$

where $S = (x)$ for $x \in |\xi - 2\mathfrak{h}|$ and $\hat{V} \in |5\xi|$ gives the proposition. □

We note that

$$F_5^2 - 4F_{10} \in |\mathcal{O}_{\mathbb{P}^3}(5)|$$

gives the section with divisor $\hat{V}$.

Computation of moduli

Proposition

The KSBA moduli space $\mathcal{M}_I$ is smooth of dimension 28.

Proof. We shall give two arguments. For the first,

$$\hat{V} \in H^0(\mathbb{P}F, 5\xi) \cong H^0(\mathbb{P}^1, S^5F).$$

Using

$$S^5F \cong \mathcal{O}_{\mathbb{P}^1}(10) \oplus \mathcal{O}_{\mathbb{P}^1}(8) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}$$

we have

$$h^0(\mathbb{P}^1, S^5F) = 11 + 9 + 7 + 5 + 3 + 1 = 36.$$

On the other hand, for $\sum_{\mathbb{P}F,\xi}$ the sheaf of differential operators on $\mathcal{O}_{\mathbb{P}F}(1)$ we have the Euler sequence

$$0 \to F^* \otimes \xi \to \Sigma_{\mathbb{P}F,\xi} \to \mathfrak{h}^2 \to 0$$

which gives

$$h^0\left(\Sigma_{\mathbb{P}F,\xi}\right) = h^0(\mathfrak{h}^2) + h^0(F^* \otimes F) = 8.$$

The argument is now a standard one.

For the alternate proof, denoting by $P_k(t_0, t_1)$ a homogeneous polynomial of degree k in t_0, t_1, we have for the weight 10 homogeneous polynomials in t_0, t_1, y, z

z^2			$y^2 P_6(t_0, t_1)$
$z \cdot P_5(t_0, t_1)$	$z \cdot P_3(t_0, t_1)$	$zy^2 P_1(t_0, t_1)$	$yP_8(t_0, t_1)$
y^5	$y^4 \cdot P_2(t_0, t_1)$	$y^3 P_4(t_0, t_1)$	$P_{10}(t_0, t_1)$

dimension $= 8 \quad +7 \quad +7 \quad +27 \quad = 49$

The automorphisms of $\mathbb{P}(1,1,2,5)$ are

$$\begin{cases} z \to zP_0 + y^2 P_1 + yP_3 + P_5 & 13 \\ y \to yP_0 + P_2 & 4 \\ t_0, t_1 \to at_0 + bt_1 + ct_0 + dt_1 & \underline{4} \\ & 21 \end{cases}$$

which gives the result. □

From Noether's formula

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + \chi_{\mathrm{top}}(X)) = \frac{1}{12}(1 + 2h^0 + 2h^{2,0} + h^{1,1})$$

we have

$$36 = 7 + h^{1,1},$$

which gives

$$h^{1,1}_{\mathrm{prim}} = 28.$$

Thus

$$\dim D = 2h^{1,1}_{\mathrm{prim}} + 1 = 57,$$

and the maximal integral manifolds of the IPR, which is a contact system, have dimension 28=number of moduli.

(iii) The local Torelli theorem for smooth I-surfaces

The argument will follow along the general lines of that for smooth surfaces in ordinary $\mathbb{P}^3$, but with an interesting wrinkle. We begin by collecting a few general facts about weighted projective spaces. For positive interers $a_0, \ldots, a_r$ with $\gcd(a_i) = 1$ we denote by

$$\mathbb{P} =: \mathbb{P}(a_0, \ldots a_r)$$

the corresponding weighted projective space, defined as the quotient

$$\mathbb{C}^{r+1} \backslash \{0\} / \mathbb{C}^*$$

by the action

$$\lambda \cdot (x_0, \ldots, x_r) = (\lambda^{a_0} x_0, \ldots \lambda^{a_r} x_r)$$

of the 1-parameter group generated by the Euler vector field

$$\mathrm{e} = \sum_{i=0}^{r} a_i x_i \partial_{x_i}.$$

For the standard sheaf $\mathcal{O}_{\mathbb{P}}(1)$ we have the weighted projective version of the consequences of Bott vanishing and Kodaira-Serre duality

- $H^0(\mathcal{O}_{\mathbb{P}}(d)) \cong$ {weighted homogeneous polynomials of degree d}
- $H^q(\mathcal{O}_{\mathbb{P}}(d)) = 0$ for all d and $0 < q < r$
- $H^r(\mathcal{O}_P(d)) \cong H^0\left(\mathcal{O}_{\mathbb{P}}\left(\sum_i a_i - d\right)\right)^*$.

We identify the sheaf $\Sigma_{\mathbb{P}}$ of differential operators of order $\leqq 1$ of $\mathcal{O}_{\mathbb{P}}(d)$ as

$$\Sigma_{\mathbb{P}} \cong \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}}(-(d - a_i))$$

via the map

$$F \to \sum_{i=0}^{r} G_i \partial_{x_i} F, \qquad G_i \in \mathcal{O}_{\mathbb{P}}(-(d - a_i)).$$

Assuming that the Jacobi ideal has no base locus, i.e., that the intersection

$$\bigcap_{i=0}^{r} \{\partial_{x_i} F = 0\} = \emptyset,$$

we have

$$\bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}}(-(d - a_i)) \xrightarrow{dF} \mathcal{O}_{\mathbb{P}} \to 0.$$

This sequence completes to a Koszul complex

$$0 \to \bigwedge^{r+1} \big((\oplus \mathcal{O}_{\mathbb{P}}(-(d - a_i))) \big) \to \bigwedge^{r} \big(\oplus \mathcal{O}_{\mathbb{P}}(-(d - a_i)) \big) \\ \to \cdots \to \mathcal{O}_{\mathbb{P}}(-(d - a_i)) \to \mathcal{O}_{\mathbb{P}} \to 0.$$

From the Bott-vanishing and duality results listed above we have

$$\frac{H^0(\mathcal{O}_{\mathbb{P}}(k))}{\operatorname{Im} dF} \cong \ker\Big\{ H^r\left(\mathcal{O}_{\mathbb{P}}(-(r+1)d + \Sigma a_i)\right) \\ \xrightarrow{dF} \oplus H^r\left(\mathcal{O}_{\mathbb{P}}(-(r+1)d + 2\Sigma a_i)\right. \Big\} \\ \cong \left(\frac{H^0(\mathcal{O}_{\mathbb{P}}(r+1)d - 2\Sigma a_i - k}{\operatorname{Im} dF}\right)^*,$$

which then gives a perfect pairing

$$\frac{H^0(\mathcal{O}_{\mathbb{P}}(k))}{\operatorname{Im} dF} \otimes \frac{H^0(\mathcal{O}_{\mathbb{P}}((r+1)d - 2\Sigma a_i - k)}{\operatorname{Im} dF} \\ \to \frac{H^0(\mathcal{O}_{\mathbb{P}}((r+1)d - 2\Sigma a_i)}{dF} \cong \mathbb{C}$$

for $0 \leqq k \leqq (r+1)d - 2\Sigma a_i$. This is the weighted projective space analogue of the usual Macaulay's theorem.

The situation we are interested in is where

- $\mathbb{P} = \mathbb{P}(a_0, a_1, a_2, a_3)$
- $F \in H^0(\mathcal{O}_{\mathbb{P}}(d))$ and $F = 0$ defines a smooth surface $X \subset \mathbb{P}$.

Then

- $K_X \cong \mathcal{O}_X(d - \Sigma a_i)$,

and for the sheaf Σ_X defined above and primitive part of $H^1(\Omega^1_X)$ we have

- $H^1(\Sigma_X) \cong H^0(\mathcal{O}_{\mathbb{P}}(d))/\operatorname{Im} dF$
- $H^1(\Omega^1_X)_{\mathrm{prim}} \cong H^0(\mathcal{O}_{\mathbb{P}}(2d - \Sigma a_i))/\operatorname{Im} dF$.

Here the second identification uses that for any smooth surface

$$H^1\left(\Omega^1_X\right)_{\mathrm{prim}} \cong H^1(\Sigma_X \otimes K_X).$$

Using the identifications

- $\bigoplus_i H^0(\mathcal{O}_{\mathbb{P}}(a_i)) \cong \bigoplus_i H^0(\mathcal{O}_X(a_i))$
- $H^0(K_X) \cong H^0(\mathcal{O}_{\mathbb{P}}(d - \Sigma a_i))$
- $H^1(\Omega^1_X)_{\mathrm{prim}} \cong \frac{H^0(\mathcal{O}_{\mathbb{P}}(2d - \Sigma a_i))}{\mathrm{Im}\, dF}$
- $H^1(\Sigma_X)^* \cong \left(\frac{H^0(\mathcal{O}_{\mathbb{P}}(3d - \Sigma a_i))}{dF}\right)^*$,

local Torelli will follow if the map

$$H^0\big(\mathcal{O}_{\mathbb{P}}(d - \Sigma a_i)\big) \otimes H^0\big(\mathcal{O}_{\mathbb{P}}(2d - \Sigma a_i)\big) \to H^0\big(\mathcal{O}_{\mathbb{P}}(3d - 2\Sigma a_i)\big)$$

is surjective. More precisely, as a consequence of the perfect pairing above

> *Local Torelli will follow if this map is surjective modulo the image of dF.*

For smooth I-surfaces we will see that the above pairing is not surjective, but it is surjective modulo the image of dF. This is in contrast to the case of the usual projective space where all maps

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(\ell)) \to H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k+\ell)), \quad k, \ell \geqq 0,$$

are surjective.
We now turn to the case where we have

$$\mathbb{P} = \mathbb{P}(1, 1, 2, 5), \qquad d = 10.$$

Then $d - \Sigma a_i = 1$ giving

$$K_X \cong \mathcal{O}_X(1),$$

and as noted above $H^0(K_X) \cong H^0(\mathbb{P}, \mathcal{O}(1))$ has dimension 2 with basis t_0, t_1.

Using a table to exhibit bases and count dimensions as was done before, for $H^0(\mathcal{O}_{\mathbb{P}}(11))$ we have

$$z^2 P^1$$

$$z \cdot y^3 \quad zj^2 P_2 \quad zyP_4 \quad zP_6$$

$$y^5 \cdot P_1 \quad y^4 P_3 \quad y^3 P_3 \quad y^2 P_7 \quad yP_9 \quad P_{11}.$$

From this we observe that

The image of the pairing $H^0(\mathcal{O}_{\mathbb{P}}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}}(10)) \to H^0(\mathcal{O}_{\mathbb{P}}(11))$ has codimension 1. In fact

$$zy^3$$

does not belong to the image.

Thus to prove local Torelli we need to show that

For a non-singular I-surface $X \subset \mathbb{P}(1,1,2,5)$ defined by the equation $F = z^2 - F_{10}(t_0, t_1, y) = 0$

$$\operatorname{Im} dF \text{ surjects onto } \frac{H^0(\mathcal{O}_{\mathbb{P}}(11))}{\operatorname{Im}\{H^0(\mathcal{O}_{\mathbb{P}}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}}(10))\}} \cong \mathbb{C}zy^3.$$

But this is clear since

$$\partial_z F = 2z + \{\text{terms not involving z}\}$$

and so that for $y^3 \in H^0(\mathcal{O}_{\mathbb{P}}(6))$ we see that $\partial_z F \cdot y^3$ is non-zero in the above quotient.

Remarks. By a suitable automorphism of $\mathbb{P}(1,1,2,5)$ we may assume that the equation of $\varphi_{2K_X}(X)$ has the above form. Calculations similar to the above give

$$\begin{aligned} T_X\mathcal{M}_I &\cong H^0(\mathcal{O}_{\mathbb{P}}(10))/\operatorname{Im} dF \text{ has dimension } = 28 \\ H^1(\Omega^1_X)_{\mathrm{prim}} &\cong H^0(\mathcal{O}_{\mathbb{P}}(11))/\operatorname{Im} dF \text{ has dimension } = 28, \end{aligned}$$

confirming our earlier computation for the first and the consequence of Noether's theorem for the second.

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