Stability of fibrations through geodesic analysis

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Motivation

Links between canonical geometric structures (differential geometry, PDEs) and stability (algebraic geometry, moduli):

Theorem (Kobayashi–Hitchin correspondence)

A holomorphic vector bundle E on a compact Kähler manifold (X, ω) admits a Hermite–Einstein metric if and only if it is slope polystable.

Conjecture (Yau–Tian–Donaldson conjecture)

Let (X, L) be a smooth polarised projective variety. Then X admits a constant scalar curvature Kähler (cscK) metric if and only if (X, L) is K-polystable.

Aim: Study the analogous conjecture for certain *fibrations*—holomorphic surjective submersions $X \to B$ whose fibres are cscK manifolds.

Outline

Main result: existence of an optimal relatively cscK metric on a fibration ("optimal symplectic connection") implies the fibration is polystable with respect to a large class of degenerations.

Key tool: geodesics in the space of all metrics and convexity of the log-norm functional (whose critical points are the optimal symplectic connections).

- Background in Kähler geometry, cscK metrics and K-stability.
- The work of Dervan and Sektnan: optimal symplectic connections and stability of fibrations
- Geodesics of Kähler metrics and their analogue in the relative setting.
- Convexity of log-norm functionals along geodesics.
- Applications to uniqueness of optimal metrics and stability of fibrations.

Kähler metrics

Let (X, ω) be a compact Kähler manifold.

The de Rham cohomology class $[\omega] \in H^2_{dR}(X; \mathbb{R})$ is called a *Kähler class*.

If $\omega' \in [\omega]$ is another Kähler metric then

$$\omega' = \omega + i\partial\bar{\partial}\phi$$

for a smooth function $\phi : X \to \mathbb{R}$ unique up to constants. We write

$$\mathcal{H} := \{\phi : X \to \mathbb{R} : \omega + i\partial\bar{\partial}\phi > 0\}$$

for the set of Kähler potentials w.r.t. ω.

Question: Is there a *canonical* or "best" Kähler metric in the class $[\omega]$?

Calabi: The best metric is a Kähler metric with constant scalar curvature.

cscK metrics

The *Ricci curvature* of ω is

$$\mathrm{Ric}(\omega) = -\frac{i}{2\pi}\partial\bar{\partial}\log\omega^n$$

and the *scalar curvature* of ω is the contraction of $\mathrm{Ric}(\omega)$ with the metric ω itself:

$$S(\omega) = \Lambda_\omega \mathrm{Ric}(\omega) = \frac{1}{2\pi}\Delta_\omega \log\omega^n.$$

A Kähler metric ω with $S(\omega)$ constant is called a *constant scalar curvature Kähler (cscK) metric*.

Note: If $L \to X$ is an ample line bundle, then $c_1(L)$ is a Kähler class.

Example (Fubini–Study metric)

Let (V, h) be a hermitian vector space. Then $\mathbb{P}(V)$ has a canonical cscK metric; the *Fubini–Study metric* determined by h. This is in the Kähler class of the ample line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$.

Test configurations

Conjecture (Yau–Tian–Donaldson)

Let L be an ample line bundle on a compact Kähler manifold X. Then $c_1(L)$ contains a cscK metric if and only if (X, L) is K-polystable.

Here K-polystability is a purely algebro-geometric condition (!) defined in terms of test configurations of the variety.

Definition

Let (X, L) be a polarised variety. A *test configuration* $(\mathcal{X}, \mathcal{L})$ for (X, L) consists of:

- A normal variety $\mathcal{X}$ with a $\mathbb{C}^*$-equivariant flat projective morphism $\mathcal{X} \to \mathbb{C}$, and
- a $\mathbb{C}^*$-equivariant relatively ample line bundle $\mathcal{L} \to \mathcal{X}$,

such that $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, L^r)$ for some $r > 0$.

K-stability

Note that $(\mathcal{X}, \mathcal{L})|_{\mathbb{C}^*} \cong (X \times \mathbb{C}^*, p^* L^r)$, so we may extend $(\mathcal{X}, \mathcal{L})$ trivially over $\infty \in \mathbb{P}^1$ to get a compact variety $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$.

The *Donaldson–Futaki invariant* of $(\mathcal{X}, \mathcal{L})$ is

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) := \frac{n}{n+1}\mu(X, L)\overline{\mathcal{L}}^{n+1} + \overline{\mathcal{L}}^{n} \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1}.$$

Definition

(X, L) is:

- *K-semistable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations $(\mathcal{X}, \mathcal{L})$ for (X, L),
- *K-polystable* if it is K-semistable, and whenever $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ we have $(\mathcal{X}, \mathcal{L}) \cong (X \times \mathbb{C}, p^* L)$ with a possibly non-trivial $\mathbb{C}^*$-action.

(Aside) Holomorphy potentials

Definition

A smooth function $f : X \to \mathbb{R}$ is a *holomorphy potential* (w.r.t. ω) if

$$\mathcal{D}f := \bar{\partial}\nabla^{1,0} f = 0,$$

i.e. $\nabla^{1,0} f$ is a holomorphic vector field.

The vector space of holomorphy potentials is finite dimensional; write E for the space of holomorphy potentials f s.t. $\int_X f\,\omega^n = 0$.

Remark

If ω is cscK, the space of cscK metrics in $[\omega]$ is a non-positively curved symmetric space

$$G/K := \mathrm{Aut}_0(X)/\mathrm{Isom}_0(\omega).$$

The tangent space to $\omega' \in G/K$ is the space $E_{\omega'}$ of holomorphy potentials w.r.t. ω'.

Set-up: relatively cscK metrics

We consider fibrations $\pi : (X, H) \to (B, L)$, where:

- X and B are compact complex manifolds,
- $\pi : X \to B$ is a holomorphic surjective submersion,
- $L \to B$ is an ample line bundle, and
- $H \to X$ is a relatively ample line bundle.

In addition, assume:

- Every fibre X_b admits a cscK metric in $c_1(H_b)$, and
- $\dim H^0(X_b, TX_b)$ is independent of b.

Under these assumptions, there exists a real closed $(1,1)$-form $\omega_X \in c_1(H)$ on X that restricts to a cscK metric ω_b on each fibre X_b. We call ω_X a *relatively cscK metric*.

Example

Let (V, h) be a hermitian vector bundle on B. Then the metric h determines a relatively cscK metric ω_h on the projectivisation $X := \mathbb{P}(V)$. Here ω_h is the curvature of the metric on $H := \mathcal{O}_{\mathbb{P}(V)}(1)$ arising from h.

Relatively cscK potentials

Define the space of *relatively cscK potentials:*

$$\mathcal{K}_E := \{\varphi \in C^\infty(X) : \omega_X + i\partial\bar{\partial}\varphi \text{ is relatively cscK}\}.$$

A relatively cscK metric ω_X induces an L^2-inner product

$$\langle f, g\rangle := \int_X fg\, \omega_X^m \wedge \omega_B^n$$

giving an orthogonal decomposition:

$$C^\infty(X) = C^\infty(B) \oplus C_E^\infty(X, \omega_X) \oplus C_R^\infty(X, \omega_X).$$

Here $f \in C_E^\infty(X, \omega_X)$ if and only if $f|_{X_b}$ is a holomorphy potential on (X_b, ω_b) with integral zero for all $b \in B$.

Lemma (Dervan–Sektnan)

If $\varphi_t : [0,1] \to \mathcal{K}_E$ is a smooth path, then

$$\dot{\varphi}_t \in C^\infty(B) \oplus C_E^\infty(X, \omega_{X,t}),$$

where $\omega_{X,t} := \omega_X + i\partial\bar{\partial}\varphi_t$.

Optimal symplectic connections

Want a canonical choice of relatively cscK metric. Fix a base Kähler metric $\omega_B \in c_1(L)$ on B and write $\omega_k := \omega_X + k\omega_B$, which is a Kähler metric on X for $k \gg 0$.

Theorem (Dervan–Sektnan)

There exists $\psi_R \in C^\infty_R(X, \omega_X)$ *such that*

$$S(\omega_k + k^{-1} i\partial\bar\partial\psi_R) = S_{\mathcal{V}}(\omega_X) + k^{-1}(S(\omega_B) + p(\theta)) + O(k^{-2}).$$

Here $p : C^\infty(X) \to C^\infty_E(X, \omega_X)$ *is the* L^2*-orthogonal projection, and*

$$\theta := \Delta_{\mathcal{V}} \Lambda_{\omega_B} \mu^* F_{\mathcal{H}} + \Lambda_{\omega_B} \rho_{\mathcal{H}}.$$

If $p(\theta) = 0$ *and* $S(\omega_B) = 0$*, can perturb* $\omega_k + k^{-1} i\partial\bar\partial\psi_R$ *to a cscK metric for* $k \gg 0$ *(assuming discrete automorphism groups).*

Definition

The relatively cscK metric ω_X is an *optimal symplectic connection* if $p(\theta) = 0$.

Fibration degenerations

Definition

Let $\pi : (X, H) \to (B, L)$ be a fibration. A *fibration degeneration* $(\mathcal{X}, \mathcal{H})$ of (X, H) consists of:

1. a normal variety $\mathcal{X}$ together with a projective morphism $\mathcal{X} \to B \times \mathbb{C}$ that has connected fibres, is flat over $\mathbb{C}$, and $\mathbb{C}^*$-equivariant over $\mathbb{C}$,
2. a $\mathbb{C}^*$-equivariant line bundle $\mathcal{H} \to \mathcal{X}$ that is relatively ample over $B \times \mathbb{C}$,
3. an isomorphism $(\mathcal{X}_1, \mathcal{H}_1) \cong (X, H^r)$ as fibrations over B.

An example to keep in mind:

Example

If $W \subset V$ is a coherent subsheaf of a holomorphic vector bundle $V \to B$, there is a degeneration of V to $W \oplus (V/W)$ inducing a fibration degeneration of $\mathbb{P}(V)$ to $\mathbb{P}(W \oplus (V/W))$.

Stability of fibrations

Given a fibration degeneration $(\mathcal{X}, \mathcal{H})$ of (X, H), there is an expansion

$$\mathrm{DF}(\mathcal{X}, kL + \mathcal{H}) = k^n W_0(\mathcal{X}, \mathcal{H}) + k^{n-1} W_1(\mathcal{X}, \mathcal{H}) + O(k^{n-2}).$$

Definition

A fibration $(X, H) \to (B, L)$ is:

- *semistable* if $W_0(\mathcal{X}, \mathcal{H}) \geq 0$ for all fibration degenerations and $W_1(\mathcal{X}, \mathcal{H}) \geq 0$ whenever $W_0(\mathcal{X}, \mathcal{H}) = 0$,
- *polystable* if it is semistable and whenever $W_0(\mathcal{X}, \mathcal{H}) = W_1(\mathcal{X}, \mathcal{H}) = 0$, the degeneration $(\mathcal{X}, \mathcal{H})$ normalises to a product fibration degeneration (or has zero norm).

Proposition (Dervan–Sektnan)

For general $b \in B$, $W_0(\mathcal{X}, \mathcal{H}) = \binom{m+n}{n} (L^n)_B \, DF(\mathcal{X}_b, \mathcal{H}_b)$.

Since (X_b, H_b) is K-polystable, we have $\mathrm{DF}(\mathcal{X}_b, \mathcal{H}_b) \geq 0$ and equality implies the general $(\mathcal{X}_b, \mathcal{H}_b)$ normalises to a product test configuration for (X_b, H_b).

Results

The central conjecture is:

Conjecture (Dervan–Sektnan)

X admits an optimal symplectic connection if and only if it is polystable.

Results towards this are:

Theorem (Dervan–Sektnan)

If $(X, H) \to (B, L)$ admits an optimal symplectic connection then it is semistable.

Theorem (H.)

If $(X, H) \to (B, L)$ admits an optimal symplectic connection then it is polystable with respect to fibration degenerations whose fibres $(\mathcal{X}_b, \mathcal{H}_b)$ are all product test configurations.

Geodesics (I)

Let (X, ω) be a compact Kähler manifold, $\mathcal{H}$ the space of Kähler potentials with respect to ω:

$$\mathcal{H} = \{\phi \in C^\infty(X) : \omega + i\partial\bar{\partial}\phi > 0\}.$$

If $\phi_t : [0,1] \to \mathcal{H}$ is a smooth path, the *energy* of ϕ is:

$$E(\phi) := \int_0^1 \int_X \dot{\phi}_t^2 \, \omega_t^n \, dt.$$

A *geodesic* is a critical point for the energy functional E.

Proposition

A smooth path $\phi_t : [0,1] \to \mathcal{H}$ is a geodesic if and only if

$$\ddot{\phi}_t = |\partial\dot{\phi}_t|^2.$$

Geodesics (II)

Remarks:

- Geodesics exist, but only in a weak ($C^{1,1}$-regular) sense.
- This is equivalent to a degenerate Monge–Ampère equation.

A useful class of smooth geodesics is furnished by holomorphic vector fields/holomorphy potentials:

Proposition

Let v be a holomorphic vector field on X, $\rho_t : X \to X$ its time t flow. Write

$$\rho_t^* \omega - \omega = i\partial\bar{\partial}\phi_t,$$

where $\int_X \dot{\phi}_t \, \omega_t^n = 0$. Then ϕ_t is a geodesic.

If $f \in E$ is a non-zero holomorphy potential, then $v := \nabla^{1,0} f$ generates a non-trivial geodesic ϕ_t in $\mathcal{H}$ with $\dot{\phi}_0 = f$.

The Mabuchi functional

Theorem (Mabuchi)

There exists a unique functional $\mathcal{M} : \mathcal{H} \to \mathbb{R}$ satisfying

$$\frac{d}{dt}\mathcal{M}(\phi_t) = \int_X \dot{\phi}_t(\hat{S} - S(\omega_t))\,\omega_t^n$$

for any smooth path $\phi_t : [0,1] \to \mathcal{H}$, and $\mathcal{M}(0) = 0$.

The functional $\mathcal{M}$ is called the *Mabuchi functional*. Its critical points are exactly the cscK metrics.

The second derivative of the Mabuchi functional along a path is

$$\frac{d^2}{dt^2}\mathcal{M}(\phi_t) = \int_X |\mathcal{D}_t\dot{\phi}_t|^2\,\omega_t^n - \int_X (\ddot{\phi}_t - |\partial\dot{\phi}_t|^2)\,\omega_t^n.$$

It follows the Mabuchi functional is convex along (smooth) geodesics in $\mathcal{H}$. It is strictly convex unless ϕ_t arises from a holomorphic vector field $\nabla^{1,0}\dot{\phi}_0$.

Uniqueness of cscK metrics

Theorem (Berman–Berndtsson)

Let ω and ω' be two cscK metrics in the same Kähler class. There exists a holomorphic automorphism $g \in Aut_0(X)$ such that $\omega' = g^*\omega$.

Suppose the Mabuchi functional $\mathcal{M}$ is convex along weak geodesics.

If ϕ_0 and ϕ_1 are two cscK potentials, there exists a unique weak geodesic ϕ_t joining them.

Then $\mathcal{M}(\phi_t)$ is convex, and $\frac{d}{dt}\mathcal{M}(\phi_t)|_{t=0} = \frac{d}{dt}\mathcal{M}(\phi_t)|_{t=1} = 0$. It follows that $\frac{d^2}{dt^2}\mathcal{M}(\phi_t) = 0$ for all $t \in [0,1]$, and in particular $\mathcal{D}\dot{\phi}_0 = 0$.

Hence the ϕ_t are generated by the holomorphic vector field

$$v := \nabla^{1,0}\dot{\phi}_0.$$

Letting ρ_t be the flow along v, we have $\rho_1^*\omega_1 = \omega_0$, and so cscK metrics are unique up to $\mathrm{Aut}_0(X)$.

K-polystability of cscK manifolds

Theorem (Berman–Darvas–Lu)

Let (X, L) be a cscK manifold. Then (X, L) is K-polystable.

Let $(\mathcal{X}, \mathcal{L})$ be a test configuration for (X, L).

Idea: associate a geodesic ray $\phi_t : [0, \infty) \to \mathcal{H}$ to $(\mathcal{X}, \mathcal{L})$ such that

$$\Phi(x, \tau) := \phi_{-\log|\tau|}(x)$$

on $\mathcal{X}|_{D^\times} \cong X \times D^\times$ extends over $\mathcal{X}|_D$.

One then calculates

$$\lim_{t \to \infty} \frac{\mathcal{M}(\phi_t)}{t} = \mathrm{DF}(\mathcal{X}, \mathcal{L}).$$

Since $\mathcal{M}$ is convex along geodesics, $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ with equality if and only if ϕ_t arises from a holomorphic automorphism, or equivalently $(\mathcal{X}, \mathcal{L})$ normalises to a product configuration.

Geodesics in the relatively Kähler setting

Let $(X, H) \to (B, L)$ be a fibration, ω_X a relatively cscK metric, and $\mathcal{K}_E$ the space of relatively cscK potentials.

There is a natural energy functional on paths $\varphi_t : [0,1] \to \mathcal{K}_E$ given by

$$E(\varphi) := \int_0^1 \int_X \dot{\varphi}_t^2 \, \omega_{X,t}^m \wedge \omega_B^n \, dt,$$

where $\omega_{X,t} := \omega_X + i\partial\bar{\partial}\varphi_t$.

φ_t is a *geodesic* if it is critical for E.

Proposition (H.)

A smooth path φ_t is a geodesic if and only if

$$\ddot{\varphi}_t = |\partial_{\mathcal{V}} \dot{\varphi}_t|^2_{\mathcal{V},t}$$

where $| \cdot |_{\mathcal{V},t}$ is the hermitian metric on $\Lambda^{1,0}\mathcal{V}^*$ determined by $\omega_{X,t}$.

This is just the usual geodesic equation but fibrewise!

Existence of geodesics

Theorem (H.)

Let $\varphi_0, \varphi_1 \in \mathcal{K}_E$. Then there exists a unique smooth geodesic $\varphi_t : [0,1] \to \mathcal{K}_E$ joining φ_0 to φ_1.

Know that smooth geodesics between cscK metrics exist, so have fibrewise existence. One only needs to show these fibrewise geodesics piece to a smooth geodesic.

To show smoothness:

- Construct a fibre bundle G/K over B whose fibre is $G_b/K_b = \mathrm{Aut}_0(X_b)/\mathrm{Isom}(\omega_b)$,
- Show this bundle embeds into the Fréchet bundle of fibrewise Kähler potentials.

A smooth choice of fibrewise cscK metrics then determines a smooth section of G/K.

Structure of the space $\mathcal{K}_E$

The proof of existence of geodesics gives:

Theorem (H.)

There is a Riemannian splitting

$$\mathcal{K}_E \cong C^\infty(B) \times C^\infty(G/K)$$

integrating the tangent space decomposition

$$T_\varphi \mathcal{K}_E = C^\infty(B) \oplus C^\infty_E(X, \omega_{X,\varphi}).$$

The smooth functions on B play the role that constant functions play in the Kähler theory.

$C^\infty(B) \subset \mathcal{K}_E$ is flat, and $C^\infty(G/K)$ is non-positively curved.

The log-norm functional for fibrations

Proposition (Dervan–Sektnan)

There exists a unique functional $\mathcal{N} : \mathcal{K}_E \to \mathbb{R}$ *such that*

$$\frac{d}{dt}\mathcal{N}(\varphi_t) = -\int_X \dot{\varphi}_t p_t(\theta_t)\, \omega_{X,t}^m \wedge \omega_B^n$$

for any smooth path $\varphi_t : [0,1] \to \mathcal{K}_E$, *and* $\mathcal{N}(0) = 0$.

The critical points of $\mathcal{N}$ are precisely the optimal symplectic connections.

Proposition (H.)

The functional $\mathcal{N}$ *is convex along smooth geodesics* φ_t *in* $\mathcal{K}_E$. *It is strictly convex, unless* φ_t *arises from a holomorphic vertical vector field.*

Aside: An automorphism of the fibration $\pi : X \to B$ is a biholomorphism $g : X \to X$ such that $\pi \circ g = \pi$. This is a complex Lie group, whose Lie algebra is the vertical holomorphic vector fields.

Convexity of $\mathcal{N}$ along geodesics

For $\psi \in C^\infty_E(X)$, define

$$\mathcal{R}\psi := \bar\partial \nabla^{1,0}_{\mathcal{V}} \psi.$$

Note the restriction of $\nabla^{1,0}_{\mathcal{V}}\psi$ to any fibre is a holomorphic vector field, and $\mathcal{R}\psi = 0$ only if the fibrewise holomorphic vector fields piece to a globally holomorphic vector field.

One computes:

$$\frac{d^2}{dt^2}\mathcal{N}(\varphi_t) = \int_X |\mathcal{R}_t \dot\varphi_t|^2_t \, \omega^m_{X,t} \wedge \omega^n_B - \int_X (\ddot\varphi_t - |\partial_{\mathcal{V}} \dot\varphi_t|^2_{\mathcal{V},t})\, \omega^m_{X,t} \wedge \omega^n_B.$$

So we have convexity along geodesics, strict unless $\mathcal{R}_t\dot\varphi_t = 0$ for all t, which is equivalent to φ_t arising from a vertical holomorphic vector field $\nabla^{1,0}_{\mathcal{V}}\dot\varphi_0$.

Uniqueness of optimal symplectic connections

Theorem (Dervan–Sektnan)

Let ω_X and ω'_X be optimal symplectic connections in same cohomology class. Then there exists a holomorphic automorphism $\rho : X \to X$ preserving π, and a smooth function $f \in C^\infty(B)$ such that

$$\omega'_X = \rho^* \omega_X + i\partial\bar{\partial} f.$$

Proof.

If $\varphi_0, \varphi_1 \in \mathcal{K}_E$ define optimal symplectic connections, let φ_t be the unique geodesic connecting them. Then

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{N}(\varphi_t) = \left.\frac{d}{dt}\right|_{t=1} \mathcal{N}(\varphi_t) = 0.$$

By convexity, $\mathcal{N}(\varphi_t)$ is constant in t, and so $\mathcal{R}\dot{\varphi}_0 = 0$. Letting ρ_t be the flow of $\nabla^{1,0}_{\mathcal{V}} \dot{\varphi}_0$, we have $\rho_1^* \omega_{X,1} + i\partial\bar{\partial} f = \omega_{X,0}$, where f is the $C^\infty(B)$-component of $\dot{\varphi}_0$. □

Polystability

Let $(\mathcal{X}, \mathcal{H})$ be a fibration degeneration for (X, H) such that every fibre $(\mathcal{X}_b, \mathcal{H}_b)$ is a product test configuration for (X_b, H_b). We want to associate a smooth geodesic ray $\varphi_t : [0, \infty) \to \mathcal{K}_E$ to $(\mathcal{X}, \mathcal{H})$.[1]

Again, we construct the fibrewise geodesics and show they depend smoothly on the base point.

Next, want to show that

$$\lim_{t \to \infty} \frac{\mathcal{N}(\varphi_t)}{t} = W_1(\mathcal{X}, \mathcal{H}).$$

To calculate this, need an analogue of the Chen–Tian formula for the Mabuchi functional.

[1] We want the S^1-invariant function $\Phi(x, \tau) := \varphi_{-\log|\tau|}(x)$ on $X \times D^\times \cong \mathcal{X}|_{D^\times}$ to extend smoothly over $\mathcal{X}|_D$.

A Chen–Tian formula (I)

Proposition (Chen–Tian formula)

$$\mathcal{M}(\phi) = \int_M \log\left(\frac{\omega_\phi^n}{\omega^n}\right)\omega_\phi^n + \sum_{i=0}^{n-1}\int_M \phi\, Ric(\omega)\wedge\omega^{n-1-i}\wedge\omega_\phi^i + \frac{\hat{S}}{n+1}\sum_{i=0}^{n}\int_M \phi\,\omega^{n-i}\wedge\omega_\phi^i.$$

Dervan–Sektnan show that the Mabuchi functional $\mathcal{M}_k$ for $\omega_X + k\omega_B$ expands as

$$\mathcal{M}_k(\varphi) = k^n\mathcal{F}(\varphi) + k^{n-1}\mathcal{N}(\varphi) + O(k^{n-2}).$$

Expanding the Chen–Tian formula, we prove:

A Chen–Tian formula (II)

Proposition (H.)

$$\mathcal{N}(\varphi) = \binom{m+n}{n-1}\Bigg[\int_X \log\left(\frac{\omega_B^n \wedge \omega_{X,\varphi}^m}{\omega_B^n \wedge \omega_X^m}\right) \omega_B^{n-1} \wedge \omega_{X,\varphi}^{m+1}$$

$$+ \sum_{i=0}^{m} \int_X \varphi\,(\rho + Ric(\omega_B)) \wedge \omega_B^{n-1} \wedge \omega_X^{m-i} \wedge \omega_{X,\varphi}^i$$

$$+ B_0 \sum_{i=0}^{m+1} \int_X \varphi\, \omega_B^{n-1} \wedge \omega_X^i \wedge \omega_{X,\varphi}^{m+1-i}$$

$$+ B_1 \sum_{i=0}^{m} \int_X \varphi\, \omega_B^n \wedge \omega_X^i \wedge \omega_{X,\varphi}^{m-i}\Bigg],$$

where ρ is the curvature of the metric on $\Lambda^m \mathcal{V}$ determined by ω_X, and the B_i are constants.

Proof of polystability (I)

The proof of polystability uses the theory of Deligne pairings, and a theorem of Boucksom–Hisamoto–Jonsson.

The Chen–Tian style formula for $\mathcal{N}$ can be written in terms of Deligne pairings of metrics on line bundles. E.g.

$$\int_X \log\left(\frac{\omega_{X,\varphi}^m \wedge \omega_B^n}{\omega_X^m \wedge \omega_B^n}\right)\omega_{X,\varphi}^{m+1} \wedge \omega_B^{n-1} = \langle \eta_{X,\varphi}, h_B^{n-1}, h_{X,\varphi}^{m+1}\rangle_X - \langle \eta_X, h_B^{n-1}, h_{X,\varphi}^{m+1}\rangle_X,$$

where η is a hermitian metric on $-K_X$, h_X a metric on $H \to X$, and h_B a metric on $L \to B$.

Theorem (Boucksom–Hisamoto–Jonsson)

$$\langle h_0^t, \dots, h_n^t\rangle_X - \langle h_0, \dots, h_n\rangle_X = t(\mathcal{L}_0 \cdot \dots \cdot \mathcal{L}_n)_{\overline{\mathcal{X}}} + O(1)$$

for line bundles $\mathcal{L}_0, \dots, \mathcal{L}_n$ on a test configuration $\mathcal{X}$ over $\mathbb{P}^1$, and h_i^t are rays of hermitian metrics on L_i compatible with $(\mathcal{X}, \mathcal{L}_i)$.

Proof of polystability (II)

We apply the Boucksom–Hisamoto–Jonsson theorem to the geodesic ray $\varphi_t : [0, \infty) \to \mathcal{K}_E$ associated to a fibration degeneration with product fibres.

For example,

$$\sum_{i=0}^{m} \int_X \varphi_t \, \omega_B^n \wedge \omega_{X,t}^i \wedge \omega_X^{m-i} = \langle \omega_B^n, \omega_{X,t}^{m+1} \rangle_X - \langle \omega_B^n, \omega_X^{m+1} \rangle_X$$

$$= t(L^n \cdot \overline{\mathcal{H}}^{m+1})_{\overline{\mathcal{X}}} + O(1).$$

Comparing the Chen–Tian style formula to the intersection theory formula for $W_1(\mathcal{X}, \mathcal{H})$, we get

$$\lim_{t \to \infty} \frac{\mathcal{N}(\varphi_t)}{t} = W_1(\mathcal{X}, \mathcal{H}).$$

Applying convexity of $\mathcal{N}$, we get polystability.