μ-cscK metric and μK-stability of polarized manifolds

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Contents:

1. Introduction to μ-cscK – special features

μ-scalar curvature: definition

$X \circlearrowleft T \cong (U(1))^{\times k}$: holomorphic action on a complex (Kähler) manifold

μ-scalar curvature

For $\lambda \in \mathbb{R}$ and $\xi \in \mathfrak{t}$ and a T-equivariant Kähler metric $\omega + \mu$, we put

$$\begin{aligned} s^\lambda_\xi(\omega) &:= (s(\omega) - \Delta\mu_\xi) - (\Delta\mu_\xi + 2|\nabla\mu_\xi|^2) + 2\lambda\mu_\xi \\ &= (s(\omega) + \bar{\Box}\theta_\xi) + (\bar{\Box}\theta_\xi - (J\xi)\theta_\xi) - \lambda\theta_\xi. \end{aligned}$$

Definition

A Kähler metric ω is a μ^λ_ξ-cscK metric if $s^\lambda_\xi(\omega)$ is constant.

- Independent of the choice of the moment map μ for ω.
- μ^λ_0-cscK metric $\iff$ cscK metric.
- When $\omega \in 2\pi\lambda^{-1}c_1(X)$,
  μ^λ_ξ-cscK metric $\iff$ Kähler-Ricci soliton: $\mathrm{Ric}(\omega) - L_{J\xi}\omega = \lambda\omega$.

μ-scalar curvature: "naturality" of the concept

Recall

Donaldson-Fujiki moment map picture

(M, ω): C^∞-symplectic manifold. Scalar curvature gives a moment map on $\mathcal{J}(M, \omega)$. Namely, the map $\mathcal{S} : \mathcal{J}(M, \omega) \to \mathrm{Lie}(\mathrm{Ham}(M, \omega))^\vee$ given by

$$\langle \mathcal{S}(J), f \rangle = \int_M (s(g_J) - \bar{s}) f \omega^n$$

is a moment map for the symplectic structure Ω on $\mathcal{J}(M, \omega)$:

$$\Omega_J(A, B) = \int_M (JA, B)_{g_J} \omega^n.$$

μ-scalar curvature: "naturality" of the concept

Put

$$\bar{s}^{\lambda}_{\xi} := \int_M s^{\lambda}_{\xi}(g_J)\ e^{\theta_\xi}\omega^n \Big/ \int_M e^{\theta_\xi}\omega^n.$$

Proposition (Moment map picture for μ-cscK, I. '18, Lahdili '18)

$(M, \omega) \circlearrowleft T$: C^∞-symplectic manifold. μ-scalar curvature gives a moment map on $\mathcal{J}_T(M, \omega)$. Namely, the map
$\mathcal{S}^{\lambda}_{\xi} : \mathcal{J}_T(M, \omega) \to \mathrm{Lie}(\mathrm{Ham}_T(M, \omega))^\vee$ given by

$$\langle \mathcal{S}^{\lambda}_{\xi}(J), f\rangle = \int_M (s^{\lambda}_{\xi}(g_J) - \bar{s}^{\lambda}_{\xi}) f\ e^{\theta_\xi}\omega^n$$

is a moment map for the symplectic structure Ω_ξ on $\mathcal{J}_T(M, \omega)$:

$$\Omega_{\xi, J}(A, B) = \int_M (JA, B)_{g_J}\ e^{\theta_\xi}\omega^n.$$

Characterization of vector fields: μ^λ-entropy

$$\boldsymbol{\mu}^\lambda(-2\xi) := -\log \frac{\mathrm{Vol}^\lambda(-2\xi)}{(n! e^n)^\lambda}$$

$$= -\frac{\int_X (s + \bar{\square}\theta_\xi) e^{\theta_\xi} \omega^n}{\int_X e^{\theta_\xi} \omega^n} + \lambda \frac{\int_X (n + \theta_\xi) e^{\theta_\xi} \omega^n}{\int_X e^{\theta_\xi} \omega^n} - \lambda \log \int_X e^{\theta_\xi} \frac{\omega^n}{n!}$$

$$\boldsymbol{\mu}^\lambda = -\frac{\int_X (\mathrm{Ric} + \bar{\square}\mu) e^{\omega+\mu}}{\int_X e^{\omega+\mu}} + \lambda \frac{\int_X (\omega + \mu) e^{\omega+\mu}}{\int_X e^{\omega+\mu}} - \lambda \log \int_X e^{\omega+\mu}$$

The functional $\boldsymbol{\mu}^\lambda$ depends only on $[\omega]$.

Proposition: μ^λ-entropy/μ^λ_ξ-Futaki invariant (I. '19)

$\exists$ μ^λ_ξ-cscK metric in $[\omega] \Rightarrow \xi$ is a critical point of $\boldsymbol{\mu}^\lambda_{[\omega]}$.

Properties of μ^λ-entropy

Proposition (I. '19)

- (Existence) Critical points of $\boldsymbol{\mu}^\lambda$ always exist regardless of the existence of μ_ξ^λ-cscK metrics in $[\omega]$.
- (Uniqueness/phase transition) For each $X \circlearrowleft T$,

$$\lambda_{\mathrm{freeze}} := \sup \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \boldsymbol{\mu}^{\lambda'} \text{ admits a unique} \\ \text{critical point for every } \lambda' \leq \lambda \end{array} \right\}$$

is always **finite** (never $\pm\infty$).

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  is always **finite** (never $\pm\infty$).
- (Extremal limit) Let ξ^λ be the unique critical point of $\boldsymbol{\mu}^\lambda$ for $\lambda < \lambda_{\mathrm{freeze}}$. Then $\lambda \xi^\lambda$ converges to the extremal vector field ξ_{ext} as λ tends to $-\infty$.

The extremal vector field ξ_{ext} is the unique critical point of

$$\int_X (\hat{s}(\omega) - \hat{\theta}_\xi)^2 \omega^n - \int_X \hat{s}^2 \omega^n. \quad (\hat{f} := f - \int_X f\omega^n / \int_X \omega^n)$$

Behavior of μ^λ-entropy: typical example

We can explicitly compute $\boldsymbol{\mu}^\lambda$ of $\mathbb{C}P^1 \circlearrowleft U(1)$. For $\xi = x.\eta \in \mathrm{u}(1)$,

$$\boldsymbol{\mu}^\lambda_{-K_{\mathbb{C}P^1}}(\xi) = 2\pi(1 - \frac{x}{\tanh x}) + \lambda(-1 + \frac{x}{\tanh x}) - \lambda \log \frac{2 \sinh x}{x}.$$

- $\lambda_{\mathrm{freeze}}(\mathbb{C}P^1, -K_{\mathbb{C}P^1}) = 4\pi$.

Behavior of μ^λ-entropy: typical example

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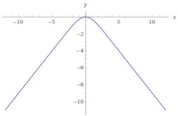
- $\lambda_{\mathrm{freeze}}(\mathbb{C}P^1, -K_{\mathbb{C}P^1}) = 4\pi$.
- There actually exists a μ^λ_ξ-cscK metric for exactly two $\xi \neq 0$ (and $\xi = 0$) when $\lambda > 4\pi$.
- ~~As $\lambda \to \infty$, the family of (non-cscK) μ^λ-cscK metrics converges on $\mathbb{C} \in \mathbb{C}P^1$, which looks like parabolic antenna~~.
- As $\lambda \to \infty$, the family of (non-cscK) μ^λ-cscK metrics ω_λ admits a family of diffeomorphisms $f_\lambda : D^2 \to \mathbb{C} \subset \mathbb{C}P^1$ from a disk of radius $\sqrt{2}$ such that $f_\lambda^* \omega_\lambda$ converges to the flat metric. (while f_λ does not converge to a diffeomorphism onto $\mathbb{C}$.)

??

Graphs of $\mu^{\lambda}(x.\ \eta)/\,2\pi$

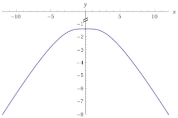
plot $1 - \frac{x}{\tanh(x)}$ $\quad \lambda = 0$



(x from −12 to 12)

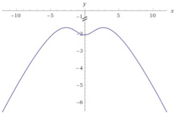
−∞
λ
+∞

plot $-1 + \frac{x}{\tanh(x)} - 2\log\left(\frac{2\sinh(x)}{x}\right)$ $\quad \lambda_{\mathrm{freeze}} = 4\pi$



(x from −12 to 12)

plot $-2 + 2 \times \frac{x}{\tanh(x)} - 3\log\left(\frac{2\sinh(x)}{x}\right)$ $\quad \lambda = 6\pi$



(x from −12 to 12)

Closedness of framework

- (Scaling) ω: μ^λ_ξ-cscK metric $\Rightarrow$ $c^{-1}\omega$: $\mu^{c\lambda}_{c\xi}$-cscK metric.
- (Product) (X, ω_X), (Y, ω_Y): μ^λ-cscK metrics with the same λ and with respect to vector fields ξ_X, ξ_Y, respectively $\Rightarrow$ $(X \times Y, \omega_X \oplus \omega_Y)$: μ^λ-cscK metric with respect to $\xi_X \oplus \xi_Y$.

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- (Perturbation of λ) $\exists$ μ^{λ}-cscK metric in $[\omega]$ with $\lambda < \lambda_1$ for the first eigenvalue λ_1 of $\Delta - \nabla \mu_{\xi}$ $\Rightarrow$ $\exists$ $\mu^{\tilde{\lambda}}$-cscK metric in the same $[\omega]$ for $\tilde{\lambda} \in (\lambda - \epsilon, \lambda + \epsilon)$.
- (Perturbation of Kähler class) We can also perturb Kähler classes under the above condition.

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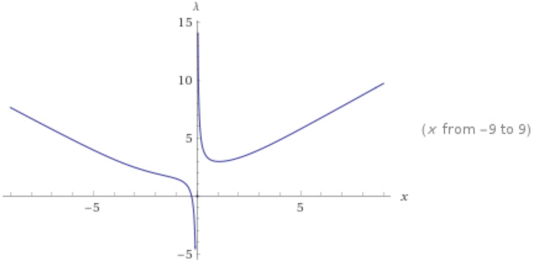
- (Perturbation of Kähler class) We can also perturb Kähler classes under the above condition.
- (Propagation) $\exists$ extremal metric in $[\omega]$ $\Rightarrow$ μ^λ-cscK metric in the same $[\omega]$ for $\lambda \ll \lambda_{\mathrm{freeze}}$ and also for $\lambda \gg \lambda_{\mathrm{freeze}}$.
- (Uniqueness) Convexity of weighted Mabuchi functional shows that μ^λ-cscK metrics are unique for $\lambda < \lambda_{\mathrm{freeze}}$. (Lahdili)

cscK $\times$ KRS (eg. toric Fano) ... ruled manifold over cscK manifold?

New!: Calabi ansatz on $\mathbb{C}P^2\#\overline{\mathbb{C}P^2} = \mathbb{P}_{\mathbb{C}P^1}(\mathcal{O}(1)\oplus\mathcal{O})$

- The anti-canonical class $-K_X$ of $X = \mathbb{C}P^2\#\overline{\mathbb{C}P^2} = \mathbb{P}_{\mathbb{C}P^1}(\mathcal{O}(1)\oplus\mathcal{O})$ admits both KRs and extremal metric (no cscK metrics).
- Calabi ansatz: $\exists$ μ^λ-cscK metrics for every $\lambda \in \mathbb{R}$ (with a negative $x_\lambda = \xi^\lambda/\eta = (6/11)\cdot\xi^\lambda/\xi_{\mathrm{ext}}$).



We can see $2.9\times 2\pi < \lambda_{\mathrm{freeze}} < 3\times 2\pi$.

2. How to formulate μK-stability? – equivariant calculus

Review on Donaldson-Futaki invariant

Recall, for a normal test configuration $(\mathcal{X}/\mathbb{C}, \mathcal{L})$ of (X, L), the Donaldson–Futaki invariant is given by

$$DF(\mathcal{X}, \mathcal{L}) := (K_{\bar{\mathcal{X}}/\mathbb{C}P^1}.\mathcal{L}^{\cdot n}) - \frac{n}{n+1}\frac{(K_X.L^{\cdot(n-1)})}{(L^{\cdot n})}(\bar{\mathcal{L}}^{\cdot(n+1)}).$$

The K-(semi)stability of (X, L) is the positivity (non-negativity) of Donaldson–Futaki invariants. (cf. Hilbert-Mumford criterion)

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- In moduli context, test configurations appear by pulling back the universal family $\mathcal{U}$ on Hilb along $\mathbb{C}^\times$-equivariant morphisms $\mathbb{C} \to \mathrm{Hilb}$, which is not necessarily normal.
- We can define DF also for non-normal $(\mathcal{X}, \mathcal{L})$ by using homology Todd class $\tau(\mathcal{O}_{\bar{\mathcal{X}}}) = [\bar{\mathcal{X}}] - \frac{1}{2}\kappa_{\bar{\mathcal{X}}} + \cdots \in A_{\mathbb{Q}}(\bar{\mathcal{X}})$ instead of $K_{\bar{\mathcal{X}}}$. (cf. Fulton, Edidin-Graham)
- The intersection formula is useful to see the behavior of $DF(\mathcal{X}, \mathcal{L})$ along the normalization and resolutions of $\mathcal{X}$.

K-stability in cscK (KE) case

Theorem (Berman-Darvas-Lu, et al.)

If the Kähler class $c_1(L)$ admits a cscK metric, then (X, L) is K-(poly)stable.

Theorem (Chen-Donaldson-Sun, Tian, (Aubin, Yau, Odaka))

The Kähler class $\lambda c_1(X)$ admits a cscK metric (KE metric) $\iff$ $(X, -\lambda K_X)$ is K-(poly)stable.

K-stability and moduli problem

Theorem (Paul-Tian)

For a G-equivariant family $(\mathcal{X}, \mathcal{L}) \to B$ of polarized schemes, there exists a G-equivariant line bundle $CM(\mathcal{X}, \mathcal{L})$ on B such that for every $\mathbb{C}^\times$-equivariant morphism $f : \mathbb{C} \to B$, the weight $-c_1^{\mathbb{C}^\times}(f^* CM(\mathcal{X}, \mathcal{L})) \in H^2_{\mathbb{C}^\times}(\mathbb{C}, \mathbb{Z}) \cong \mathbb{Z}.\eta^\vee$ is equal to $DF(f^*\mathcal{X}, f^*\mathcal{L})$.

Theorem (Odaka, Li-Wang-Xu)

$\mathbb{Q}$-smoothable Fano varieties with Kähler–Einstein metrics form a proper algebraic moduli space.

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Theorem (I. '19)

Fano manifolds with Kähler-Ricci solitons ($= \mu^{2\pi}$-cscK metric in $c_1(X)$) form a complex analytic moduli space.

Can we compactify the moduli space (or make it algebraic)?

μFutaki invariant

(X, L): T-equivariant polarized manifold (scheme)
For $\xi \in \mathfrak{t}$, we define the μ^0_ξ-Futaki invariant of a T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$ by the following equivariant intersection formula:

$$\mathrm{Fut}^0_\xi(\mathcal{X}, \mathcal{L}) := 4\pi \frac{\mathrm{Ev}_\xi\Big((\kappa^T_{\bar{\mathcal{X}}/\mathbb{C}P^1} . e^{\bar{\mathcal{L}}_T}) \cdot (e^{L_T}) - (\kappa^T_X . e^{L_T})(e^{\bar{\mathcal{L}}_T}) \Big)}{(\mathrm{Ev}_\xi(e^{L_T}))^2} \in \mathbb{R}.$$

When $\mathcal{X}$ is smooth, this is equivalent to:

$$-2\frac{\int_{\bar{\mathcal{X}}} (\mathrm{Ric}^{\mathrm{rel}}_{\tilde{\Omega}} + \bar{\Box}_{\tilde{\Omega}} \tilde{\Theta}_\xi) e^{\Omega+\Theta_\xi} \int_X e^{\Omega+\Theta_\xi} - \int_X (\mathrm{Ric}_\omega + \bar{\Box}\theta_\xi) e^{\omega+\theta_\xi} \int_{\bar{\mathcal{X}}} e^{\Omega+\Theta_\xi}}{(\int_X e^{\omega+\theta_\xi})^2},$$

where $\mathrm{Ric}^{\mathrm{rel}}_{\tilde{\Omega}} = \mathrm{Ric}(\tilde{\Omega}) - \pi^* \mathrm{Ric}(\omega_{\mathbb{C}P^1})$ for some metrics $\tilde{\Omega}, \omega_{\mathbb{C}P^1}$ on $\bar{\mathcal{X}}, \mathbb{C}P^1$.
We can similarly define

$$\mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L}) := \mathrm{Fut}^0_\xi(\mathcal{X}, \mathcal{L}) + \lambda(\text{equiv. intersection on } \bar{\mathcal{L}}).$$

μK-semistability

A T-polarized manifold is μ^λ_ξK-semistable if Fut^λ_ξ is non-negative for any test configuration.

Proposition (I. '20)

1. For smooth X, the μ^λ_ξK-semistability of (X, L) with respect to general test configurations is equivalent to the μ^λ_ξK-semistability with respect to smooth test configurations (test configurations with smooth total space $\mathcal{X}$).
2. For smooth test configuration, Fut^λ_ξ is equivalent to one of Lahdili's weighted Futaki invariants.

Cororally (Essentially, Lahdili's result on weighted cscK '19)

If a smooth T-polarized manifold (X, L) admits a μ^λ_ξ-cscK metric in $c_1(L)$, then (X, L) is μ^λ_ξK-semistable (with respect to general test configurations).

Generalization of CM line bundle

Theorem (I. '20)

For $\lambda \in \mathbb{R}$ and $\xi \in \mathfrak{t}$, there exists a characteristic class $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda$ assigning $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda(\mathcal{X}/B, \mathcal{L}) \in H^2_G(B, \mathbb{R})$ for each $T \times G$-equivariant family of polarized schemes $(\mathcal{X}/B, \mathcal{L})$ over smooth G-variety B which enjoys the following:

1. Naturality: $f^* \mathcal{D}_\xi \boldsymbol{\mu}^\lambda(\mathcal{X}/B, \mathcal{L}) = \mathcal{D}_\xi \boldsymbol{\mu}^\lambda(\mathcal{X}'/B', \mathcal{L}')$ for

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

2. μ-Futaki invariant: $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda(\mathcal{X}/\mathbb{C}, \mathcal{L}) = \mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L}).\eta^\vee$ for any T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$
3. CM line bundle: $\mathcal{D}_0 \boldsymbol{\mu}^\lambda_G(\mathcal{X}/B, \mathcal{L}) = -\frac{4\pi}{(L^{\cdot n})} c_1^G(CM(\mathcal{X}/B, \mathcal{L}))$

Application

Combining with Chen–Sun's deep analysis on Kähler–Ricci flow and the analytic openness of μK-semistable locus established in the previous work, we can show that μK-semistable locus for a polarized family is Zariski open on the base. Then we get the following result on algebraicity.

Cororally

The moduli space of Fano manifolds with Kähler–Ricci solitons is an algebraic space.

I also have a plan for compactifying the moduli space. (in progress)

Idea of construction – the case $\lambda = 0$ (to economize space)

Recall the following expression of μ^λ-entropy:

$$\boldsymbol{\mu}^0 = -\frac{\int_X (\mathrm{Ric} + \bar{\Box}\mu) e^{\omega+\mu}}{\int_X e^{\omega+\mu}}.$$

Both $\int_X (\mathrm{Ric} + \bar{\Box}\mu) e^{\omega+\mu}$ and $\int_X e^{\omega+\mu}$ are the integration of equivariant forms. In other words, we can regard these as the pushforward of the equivariant cohomology classes

$$K_X^T \frown e^{L_T}, e^{L_T} \in \hat{H}_T(X, \mathbb{R}) := \prod_{k=0}^{\infty} H_T^{2k}(X, \mathbb{R})$$

along $p : X \to \mathrm{pt}$, which are elements of $\hat{H}_T(\mathrm{pt}, \mathbb{R}) \cong \prod_{k=0}^{\infty} S^k \mathfrak{t}^\vee$ and are the Taylor expansion (at $0 \in \mathfrak{t}$) of the functionals $\int_X (\mathrm{Ric} + \bar{\Box}\mu) e^{\omega+\mu}$, $\int_X e^{\omega+\mu}$ on $\mathfrak{t}$. For a G-equivariant polarized family $(\mathcal{X}/B, \mathcal{L})$, we put

$$\boldsymbol{\mu}^0_{\mathcal{X}/B,\mathcal{L}} := 2\pi \frac{\pi_*(\kappa_{\mathcal{X}/B}.e^{\mathcal{L}})}{\pi_*(e^{\mathcal{L}})} \in \hat{H}_G(B, \mathbb{R}).$$

Idea of construction – Sketch of equivariant calculus

1. (Differential at ξ along G) For $\xi \in \mathfrak{t}$, we introduce a differential operation
$$\mathcal{D}_\xi : H^\omega_{T\times G}(B,\mathbb{R}) \to H^2_G(B,\mathbb{R})$$
for some subring $H^\omega_{T\times G}(B,\mathbb{R})$ of $\hat{H}_{T\times G}(B,\mathbb{R})$ where T acts on B trivially. When $G = \{1\}$ and $B = \mathrm{pt}$, $H^\omega_{T\times G}(B,\mathbb{R})$ is identified with the ring of real analytic functions on $\mathfrak{t}$.
2. (Convergence result) For $T \times G$-equivariant polarized family $(\mathcal{X}/B,\mathcal{L})$, we can show that $\boldsymbol{\mu}^\lambda_{\mathcal{X}/B,\mathcal{L}}$ is in $H^\omega_{T\times G}(B,\mathbb{R})$, using Cartan model of equivariant deRham current homology. The element $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{\mathcal{X}/B,\mathcal{L}} \in H^2_G(B,\mathbb{R})$ is what we want!
3. (Equivariant Grothendieck-Riemann-Roch) Naturality and the identification with CM line bundle comes from the equivariant Grothendieck-Riemann-Roch theorem by Edidin-Graham.
4. (Localization formula) Using the equivariant localization formula, we can see $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{\mathcal{X}/\mathbb{C},\mathcal{L}} = \mathrm{Fut}^\lambda_\xi(\mathcal{X},\mathcal{L}).\eta^\vee$.

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4. (Localization formula) Using the equivariant localization formula, we can see $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{\mathcal{X}/\mathbb{C}, \mathcal{L}} = \mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L}).\eta^\vee$. Thank you for listening!

Latest news: Han-Li's result when $L = -K_X$

- Berman–Witt-Nyström proved that $(X, -K_X)$ is $\mu^{2\pi}$K-polystable with respect to special degenerations if X admits a KRs ($=$ $\mu^{2\pi}$-cscK metric).
- Recently, J. Han and C. Li introduced G-uniform g-Ding stability '$\boldsymbol{D}_g^{NA}(\phi) \geq \gamma \cdot \boldsymbol{J}_g^{NA}(\phi)$' and proved the equivalence of G-uniform g-Ding stability of $(X, -K_X)$ for 'maximal' G is equivalent to the existence of KR g-soliton.
- They also show that the (G-uniform) g-Ding stability of $(X, -K_X)$ is equivalent to that with respect to special degenerations, using MMP with scaling. The proof works also for $\boldsymbol{M}_g^{NA}$. (I guess it works also for Fut_ξ^λ.)
- g-Mabuchi stability for $g = e^{\langle \xi, - \rangle}$ must be equivalent to μ_ξ^λK-stability. (λ is determined from ξ.)
- Thus, $\exists$ KRs on X $\iff$ $(X, -K_X)$ is $\mu^{2\pi}$K-polystable.