Bigerbes and applications

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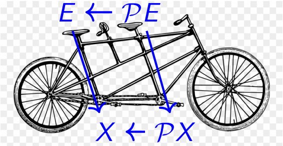
UQAM Topology and Geometry Seminar, 26 February 2021



$$\text{Line bundles} \leftrightarrow H^2(X;\mathbb{Z})$$

$$\text{Gerbes} \leftrightarrow H^3(X;\mathbb{Z})$$

$$\text{Bigerbes} \leftrightarrow H^4(X;\mathbb{Z})$$

$$n\text{-multigerbes} \leftrightarrow H^{n+2}(X;\mathbb{Z})$$

- Geometric objects which represent, and are classified by, integer cohomology
- Nontriviality corresponds to obstruction

A complex line bundle $L \longrightarrow X$ (or principal $\mathbb{C}^*$ or $\mathbb{T}$-bundle) has a Chern class $c_1(L) \in H^2(X;\mathbb{Z})$, with the following properties:

- Naturality:

$$\begin{array}{ccc} f^*L & \longrightarrow & L \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

  implies $c_1(f^*L) = f^*c_1(L) \in H^2(X';\mathbb{Z})$.
- Additivity: $c_1(L \otimes L') = c_1(L) + c_1(L')$, $c_1(L^{-1}) = -c_1(L)$.
- Triviality: $c_1(L) = 0$ if and only if L is trivial: $L \cong X \times \mathbb{C}$.
- Classification: $c_1(L) = c_1(L')$ if and only if $L \cong L'$.
- Universality: there is a bijection $H^2(X;\mathbb{Z}) \leftrightarrow \{L \longrightarrow X\} / \cong$.

Nontriviality is the obstruction to admitting a trivialization, i.e., a global nonvanishing section.

Various versions: Giraud, Brylinski, Hitchin and Chattergee, Murray.
[Murray '96] A **(bundle) gerbe** (L, Y, X) consists of:

- a *locally split* map (meaning surjective with local sections; e.g. fiber bundle)
$$\pi : Y \longrightarrow X,$$
- a line bundle (or principal $\mathbb{C}^*$ or $\mathbb{T}$-bundle)
$$L \longrightarrow Y^{[2]}, \quad Y^{[2]} = Y \times_X Y = \{(y_1, y_2) : \pi(y_1) = \pi(y_2) \in X\}$$
- with an associative product
$$\phi : L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \xrightarrow{\cong} L_{(y_1,y_3)}, \quad (y_1, y_2, y_3) \in Y^{[3]}.$$

Associativity means that

$$\phi \circ (1 \otimes \phi) = \phi \circ (\phi \otimes 1) : L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \otimes L_{(y_3,y_4)} \cong L_{(y_1,y_4)},$$
$$(y_1, y_2, y_3, y_4) \in Y^{[4]}.$$

Can say: Y is a $\mathbb{C}^*$ *groupoid over* X (each $\mathrm{Hom}(y_1, y_2) \cong \mathbb{C}^*$).

Properties of gerbes

$$\begin{array}{ccccc} & & & & L \\ & & & & \downarrow \\ X & \xleftarrow{\pi} & Y & \overset{\pi_1}{\underset{\pi_0}{\leftleftarrows}} & Y^{[2]} \end{array}$$

To a bundle gerbe (L, Y, X), we associate a **Dixmier-Douady class** $DD(L, Y, X) \in H^3(X; \mathbb{Z})$, with the following properties:

- Naturality:

$$\begin{array}{ccc} f^*L & \longrightarrow & L \\ \downarrow & & \downarrow \\ f^*Y^{[2]} & \longrightarrow & Y^{[2]} \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

  implies $DD(f^*L, f^*Y, X) = f^*DD(L, Y, X) \in H^3(X; \mathbb{Z})$.

- Additivity: $(L, Y, X) \otimes (L', Y', X) = (L \otimes L', Y \times_X Y', X)$, with respect to which

$$DD(L \otimes L', Y \times_X Y', X) = DD(L, Y, X) + DD(L', Y', X),$$
$$DD(L^{-1}, Y, X) = -DD(L, Y, X).$$

$$\begin{array}{ccccc} & & Q & & L \\ & & \downarrow & & \downarrow \\ X & \xleftarrow[\pi]{} & Y & \overset{\pi_1}{\underset{\pi_0}{\leftleftarrows}} & Y^{[2]} \end{array}$$

To a bundle gerbe (L, Y, X), we associate a **Dixmier-Douady class** $DD(L, Y, X) \in H^3(X; \mathbb{Z})$, with the following properties:

- Triviality: A **trivialization** of a bundle gerbe is a line bundle $Q \longrightarrow Y$ such that $L \cong \pi_0^* Q \otimes \pi_1^* Q^{-1} =: \delta Q$, and $DD(L, Y, X) = 0$ if and only if such a trivialization exists. (This is weaker than a trivialization of the line bundle $L \longrightarrow Y^{[2]}$.)
- Classification: $DD(L, Y, X) = DD(L', Y', X)$ if and only if (L, Y, X) and (L', Y', X) are *stably isomorphic* [Murray-Stevenson '00] (essentially, $(L, Y, X) \otimes (L', Y', X)$ is trivial).

Application: lifting bundle gerbes

Suppose a group G has a central extension by $\mathbb{T}$ (or $\mathbb{C}^*$):

$$\mathbb{T} \longrightarrow \widehat{G} \longrightarrow G.$$

In particular, $\widehat{G} \longrightarrow G$ is a $\mathbb{T}$-bundle.

If $E \longrightarrow X$ is a principal G-bundle, there is a difference map $\chi : E^{[2]} \longrightarrow G$, where $\chi(y_0, y_1) = g$ such that $y_1 = y_0\, g$.

The **lifting bundle gerbe** for E is $(\chi^* \widehat{G}, E, X)$:

$$\begin{array}{ccc} \chi^* \widehat{G} & & \widehat{G} \\ \downarrow & & \downarrow \\ E^{[2]} & \xrightarrow{\chi} & G \\ \downarrow & & \\ X & & \end{array}$$

The gerbe product comes from the group structure of $\widehat{G}$.

$$\begin{array}{ccccccc} & & \widehat{E} & & \chi^*\widehat{G} & & \widehat{G} \\ & & \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & E & \Longleftarrow & E^{[2]} & \xrightarrow{\chi} & G \end{array}$$

Theorem (Murray '96)

A trivialization of $(\chi^*\widehat{G}, E, X)$ is equivalent to a lift of E to a principal $\widehat{G}$-bundle $\widehat{E} \longrightarrow X$.

Thus $DD(\chi^*\widehat{G}, E, X) \in H^3(X;\mathbb{Z})$ is the obstruction to refining the G structure $E \longrightarrow X$ to a $\widehat{G}$-structure.

A familiar example: $G = \mathrm{SO}(n)$, $\widehat{G} = \mathrm{Spin}^c(n)$:

$$\mathbb{T} \longrightarrow \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n),$$

$E_{\mathrm{SO}} \longrightarrow X$ the frame bundle of a Riemannian manifold. Then $DD(\chi^*\mathrm{Spin}^c(n), E_{\mathrm{SO}}, X) = W_3(X) \in H^3(X;\mathbb{Z})$ is the obstruction to a spin^c structure on X.

Universal example: loop space

Fix a basepoint in X and let $\mathcal{P}X = \mathrm{Map}_*([0,1], X)$ be the based path space, with evaluation map

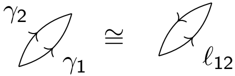
$$\varepsilon : \mathcal{P}X \longrightarrow X, \quad \gamma \mapsto \gamma(1).$$

If X locally contractible, then $\mathcal{P}X \longrightarrow X$ is locally split.

- Universality: Every class in $H^3(X;\mathbb{Z})$ is represented by a bundle gerbe $(L, \mathcal{P}X, X)$ [Brylinski, Murray].

(In general, a class is representable by a bundle gerbe (L, Y, X) if and only if it pulls back trivially to $H^3(Y;\mathbb{Z})$ [K-Melrose '19].)

Note that we can identify $\mathcal{P}^{[2]}X$ with the based *loop space* $\mathcal{L}X = \mathrm{Map}_*(S^1, X)$:



so the gerbe is a certain line bundle $L \longrightarrow \mathcal{L}X$, with $c_1(L) \in H^2(\mathcal{L}X;\mathbb{Z})$.

How is $c_1(L) \in H^2(\mathcal{L}X;\mathbb{Z})$ related to $DD(L) \in H^3(X;\mathbb{Z})$?

Universal example: loop space

Recall that there is a *transgression* $\tau : H^\bullet(X;\mathbb{Z}) \longrightarrow H^{\bullet-1}(\mathcal{L}X;\mathbb{Z})$ in cohomology, where

$$\begin{array}{ccc} H^\bullet(X;\mathbb{Z}) & \xrightarrow{\mathrm{ev}^*} & H^\bullet(S^1 \times \mathcal{L}X;\mathbb{Z}) \\ & \searrow^{\tau} & \downarrow{\cap[S^1]} \\ & & H^{\bullet-1}(\mathcal{L}X;\mathbb{Z}) \end{array}$$

The class of $c_1(L) \in H^2(\mathcal{L}X;\mathbb{Z})$ is the image under τ of $DD(L, \mathcal{P}X, X) \in H^3(X;\mathbb{Z})$. In general, τ is neither injective nor surjective. This map in cohomology forgets the gerbe product on L.

With the identification $\mathcal{P}^{[2]}X \cong \mathcal{L}X$, the gerbe product is known as *fusion* in the literature [Stolz-Teichner '04, Waldorf '12]:

$$L_{\ell_{12}} \otimes L_{\ell_{23}} \xrightarrow{\cong} L_{\ell_{13}}$$

Thus a gerbe $(L, \mathcal{P}X, X)$ is equivalent to a *fusion line bundle* $L \longrightarrow \mathcal{L}X$, and there is a bijection

$$H^3(X;\mathbb{Z}) \leftrightarrow \{\text{Fusion line bundles } L \longrightarrow \mathcal{L}X\} \,/ \sim$$

Remark

- Fusion can be built into a cohomology theory $H^\bullet_{\mathrm{fus}}(\mathcal{L}X;\mathbb{Z})$, for which $\tau : H^\bullet(X;\mathbb{Z}) \xrightarrow{\cong} H^{\bullet-1}_{\mathrm{fus}}(\mathcal{L}X;\mathbb{Z})$ is an isomorphism [K-Melrose '15].
- With an additional condition, this can all be promoted to the free loop space.

Gerbes as simplicial line bundles

We can reinterpret bundle gerbes as follows: the sequence of fiber products $Y^{[\bullet]}$ in

$$\begin{array}{ccccccccc} & & & & L & & \delta L & & \delta^2 L \\ & & & & \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & Y & \Lleftarrow & Y^{[2]} & \Lleftarrow & Y^{[3]} & \Lleftarrow & Y^{[4]} \quad \cdots\cdots \end{array}$$

forms a *simplicial space* over X, meaning the projections $\pi_j : Y^{[n]} \longrightarrow Y^{[n-1]}$, $j = 0, \ldots, n-1$ satisfy simplicial relations ($\pi_i \pi_j = \pi_{j-1} \pi_i$, $i < j$, etc).

From a line bundle $L \longrightarrow Y^{[n]}$ we can form the "differential" $\delta L = \bigotimes_{j=0}^{n} \pi_j^* L^{(-1)^j} \longrightarrow Y^{[n+1]}$, and $\delta^2 L \longrightarrow Y^{[n+2]}$ is canonically trivial.

A bundle gerbe is equivalent to a *simplicial line bundle* [Brylinski-McLaughlin '94], meaning $L \longrightarrow Y^{[2]}$ with a trivialization of δL inducing the canonical trivialization of $\delta^2 L$.

- The trivialization of δL is equivalent to the gerbe product.
- The condition on $\delta^2 L$ is equivalent to associativity.
- L is trivial if $L = \delta Q$.

[Stevenson '04]: Bundle gerbes have pullbacks, products, inverses, and trivializations, so we can play the same game again. A **bundle 2-gerbe** (L, Z, Y, X) is a "simpicial bundle gerbe"

$$\begin{array}{ccccccccc} & & & & \mathbb{L} & & \delta\mathbb{L} & & \delta^2\mathbb{L} & & \text{eek!} \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ X & \longleftarrow & Y & \xleftarrow[\times 2]{} & Y^{[2]} & \xleftarrow[\times 3]{} & Y^{[3]} & \xleftarrow[\times 4]{} & Y^{[4]} & \xleftarrow[\times 5]{} & Y^{[5]} \end{array}$$

consisting of:

- A locally split map $Y \longrightarrow X$,
- A bundle gerbe $\mathbb{L} = (L, Z, Y^{[2]})$,
- A trivialization of $\delta\mathbb{L} = \pi_0^*\mathbb{L} \otimes \pi_1^*\mathbb{L}^{-1} \otimes \pi_2^*\mathbb{L}$ over $Y^{[3]}$,
- A 2-morphism (did I mention gerbes have 2-morphisms?) relating the induced trivialization of $\delta^2\mathbb{L}$ to the canonical one,
- A coherency condition on pulled back 2-morphisms over $Y^{[5]}$.

Benefits: (L, Z, Y, X) has a well-defined characteristic class in $H^4(X;\mathbb{Z})$ with the expected properties.

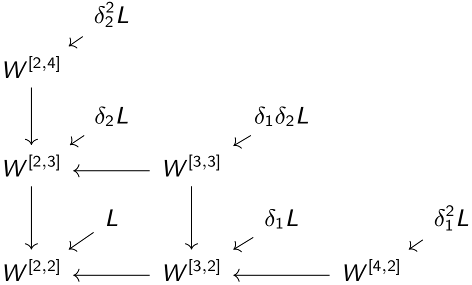
Drawbacks: The roles of Y and Z are very asymmetric. For higher gerbes ($H^{\geq 5}(X;\mathbb{Z})$), higher and more complicated coherency conditions will appear.

[K-Melrose '19] Bigerbes as an alternative version of 2-gerbes.

$$
\begin{array}{ccccccc}
Z^{[3]} & \longleftarrow & W^{[1,3]} & \Lleftarrow & W^{[2,3]} & \Lleftarrow & W^{[3,3]} \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
Z^{[2]} & \longleftarrow & W^{[1,2]} & \Lleftarrow & W^{[2,2]} & \Lleftarrow & W^{[2,3]} \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
Z & \longleftarrow & W & \Lleftarrow & W^{[2,1]} & \Lleftarrow & W^{[3,1]} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longleftarrow & Y & \Lleftarrow & Y^{[2]} & \Lleftarrow & Y^{[3]}
\end{array}
$$

The setup:

- A **locally split square** is a diagram as above where $Y \longrightarrow X$ and $Z \longrightarrow X$ and $W \longrightarrow Y \times_X Z$ are locally split.
- Fill out the diagram by fiber products.
- $W^{[\bullet,\bullet]}$ forms a *bisimplicial space over* X (each row and column is simplicial), and there are two (commuting) differentials δ_1 and δ_2 on line bundles.



A **bundle bigerbe** (L, W, Z, Y, X) is a "bisimplicial line bundle", meaning a line bundle L over $W^{[2,2]}$, with trivializations of $\delta_1 L$ and $\delta_2 L$ inducing the canonical trivializations of $\delta_1^2 L$ and $\delta_2^2 L$, and agreeing on $\delta_1\delta_2 L$.

Products, inverses, and pullbacks are straightforward to define.

A **trivialization** is an isomorphism $L \cong \delta_1 Q_1 \otimes \delta_2 Q_2$ for a line bundles $Q_1 \longrightarrow W^{[1,2]}$ and $Q_2 \longrightarrow W^{[2,1]}$.

Theorem (K-Melrose '19)

A bundle bigerbe (L, W, Z, Y, X) has a well-defined characteristic class $G(L) := G(L, W, Z, Y, X) \in H^4(X;\mathbb{Z})$, with the following properties:

- *Naturality:* $G(f^*L) = f^*G(L)$
- *Additivity:* $G(L \otimes L') = G(L) + G(L')$, $G(L^{-1}) = -G(L)$.
- *Triviality:* $G(L) = 0$ *if and only if L admits a trivialization:* $L \cong \delta_1 Q_1 \otimes \delta_2 Q_2$.
- *Classification:* $G(L) = G(L')$ *if and only if the two bigerbes are* stably isomorphic.

There is also a theorem characterizing which squares (W, Z, Y, X) support a bigerbe with a given 4-class.

Benefits: No higher category stuff! Also generalizes in a straightforward manner to higher degree, leading to **bundle multigerbes**. (Start with a locally split n-cube...)

But are there any interesting ones?

Recall the *Whitehead tower* for $G = \mathrm{O}(n) = \mathrm{O}$:

$$\mathrm{O} \longleftarrow \mathrm{SO} \longleftarrow \mathrm{Spin} \longleftarrow \mathrm{String} \longleftarrow \cdots$$

$$\pi_0 = 0 \qquad \pi_{\leq 2} = 0 \qquad \pi_{\leq 6} = 0$$

$$\pi_0 = \mathbb{Z}_2 \qquad \pi_1 = \mathbb{Z}_2 \qquad \pi_3 = \mathbb{Z}$$

The **string group** is a 3-connected (in fact 6-connected) topological group sitting over Spin.

While $\mathbb{Z}_2 \longrightarrow \mathrm{Spin} \longrightarrow \mathrm{SO}$ is a finite cover (and central extension), String is not a finite dimensional Lie group (Ker$(\mathrm{String} \longrightarrow \mathrm{Spin})$ has the homotopy type of $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$).

If X is a Riemannian manifold with frame bundle $E_{\mathrm{O}} \longrightarrow X$, then a lift to $E_G \longrightarrow X$ is:

G	**structure**	**obstruction**	**classification**
SO	orientation	$w_1(X) \in H^1(X; \mathbb{Z}_2)$	$H^0(X; \mathbb{Z}_2)$
Spin	spin	$w_2(X) \in H^2(X; \mathbb{Z}_2)$	$H^1(X; \mathbb{Z}_2)$
String	string	$\frac{1}{2} p_1(X) \in H^4(X; \mathbb{Z})$	$H^3(X; \mathbb{Z})$

If $E \longrightarrow X$ is a principal G-bundle, then $\mathcal{L}E \longrightarrow \mathcal{L}X$ is a principal $\mathcal{L}G$ bundle, and there is a degree lowering "transgression relation" between structures on X and those on $\mathcal{L}X$.

Suppose X is oriented, so $G = \mathrm{SO}$. Then $\mathcal{L}\mathrm{SO}$ is not connected; a reduction to $\mathcal{L}\mathrm{SO}_+ \cong \mathcal{L}\mathrm{Spin}$ is therefore an "orientation" on $\mathcal{L}X$, or a *loop-orientation* [Atiyah '85, McLaughlin '92].

If X admits a spin structure, then $\mathcal{L}X$ admits an orientation (not iff!).

[Stolz-Teichner '04]: There is a natural bijection between spin structures on X and *fusion orientations* on $\mathcal{L}X$.

A lift of $\mathcal{L}E_{\mathrm{Spin}} \longrightarrow \mathcal{L}X$ for a spin manifold to the universal central extension

$$\mathbb{T} \longrightarrow \widehat{\mathcal{L}\mathrm{Spin}} \longrightarrow \mathcal{L}\mathrm{Spin}$$

(determined by a generator of $H^3(\mathrm{Spin};\mathbb{Z}) = \mathbb{Z}$) is a "spin structure" on $\mathcal{L}X$ [Atiyah '85, McLaughlin '92], or a *loop-spin structure*.

The obstruction is $\tau\left(\frac{1}{2}p_1(X)\right) \in H^3(\mathcal{L}X;\mathbb{Z})$, so if X admits a string structure, then $\mathcal{L}X$ admits a loop-spin structure (not iff!).

[Stolz-Teichner '04, Waldorf '16, K-Melrose '13]: Relations between string structures on X and fusion loop-spin (and related) structures on $\mathcal{L}X$.

[Brylinski-McLaughlin, CJMSW] 2-gerbes related to the spin structure on $\mathcal{L}X$.

There is a bigerbe giving a very clear picture.

Take X a spin manifold with $E = E_{\mathrm{Spin}} \longrightarrow X$. Start with locally split square $(\mathcal{P}E, E, \mathcal{P}X, X)$.

$$
\begin{array}{ccccccc}
 & & & & & & \widehat{\mathcal{L}\mathrm{Spin}} \\
 & & & & & & \downarrow \\
E^{[2]} & \longleftarrow & \mathcal{P}E^{[2]} & \Longleftarrow & \mathcal{L}E^{[2]} & \xrightarrow{\chi} & \mathcal{L}\mathrm{Spin} \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \\
E & \longleftarrow & \mathcal{P}E & \Longleftarrow & \mathcal{L}E & & \\
\downarrow & & \downarrow & & \downarrow & & \\
X & \longleftarrow & \mathcal{P}X & \Longleftarrow & \mathcal{L}X & &
\end{array}
$$

Theorem (K-Melrose '19)

$\chi^*(\widehat{\mathcal{L}\mathrm{Spin}}) \longrightarrow \mathcal{L}E^{[2]}$ determines a bigerbe, with characteristic class $\frac{1}{2}p_1(X) \in H^4(X;\mathbb{Z})$. Trivializations of this bigerbe are equivalent to fusion loop-spin structures, and these are in bijection with $H^3(X;\mathbb{Z})$, and hence with string structures on X.

- Vertical simpliciality comes from the lifting bundle gerbe.
- Horizontal simpliciality comes from a fusion/gerbe property of $\widehat{\mathcal{L}\mathrm{Spin}} \longrightarrow \mathcal{L}\mathrm{Spin}$.

One other nice application of bigerbes is to “decomposable” classes.

A class $\alpha = \beta_1 \cup \beta_2 \in H^4(X;\mathbb{Z})$, with $\beta_i \in H^2(X;\mathbb{Z})$ is represented by a bigerbe as follows.

Let $Y_i \longrightarrow X$ be principal $\mathbb{T}$-bundles with $c_1(Y_i) = \beta_i \in H^2(X;\mathbb{Z})$, and set $W = Y_1 \times_X Y_2 = Y_1 \otimes Y_2$, a principal $\mathbb{T}^2$ bundle, to get the split square

$$\begin{array}{ccc} Y_2 & \longleftarrow & Y_1 \otimes Y_2 \\ \downarrow & & \downarrow \\ X & \longleftarrow & Y_1 \end{array}$$

Then $W^{[2,2]} = Y_1^{[2]} \otimes Y_2^{[2]} = (Y_1 \otimes Y_2)^{[2]}$ in this case, with difference map

$$\chi : (Y_1 \otimes Y_2)^{[2]} \longrightarrow \mathbb{T}^2,$$

and $\chi^* S \longrightarrow Y_1^{[2]} \otimes Y_2^{[2]}$ defines a bigerbe with class $\beta_1 \cup \beta_2 \in H^4(X;\mathbb{Z})$, where $S \longrightarrow \mathbb{T}^2$ is an explicit representative of the fundamental line bundle: $c_1(S) = 1 \in H^2(\mathbb{T}^2;\mathbb{Z}) \cong \mathbb{Z}$.

Likewise, suppose $\beta \in H^3(X;\mathbb{Z})$ and $\gamma \in H^1(X;\mathbb{Z})$, we may represent $\gamma \cup \beta$ as follows.

Suppose (L, Y, X) is a bundle gerbe with $DD(L) = \beta$.

Represent γ by a map $X \longrightarrow \mathbb{T} = K(\mathbb{Z}, 1)$, and pull back the universal cover $\mathbb{R} \longrightarrow \mathbb{T}$ to the $\mathbb{Z}$ cover $\widetilde{X} \longrightarrow X$.

Form the locally split square

$$\begin{array}{ccc} Y & \longleftarrow & \widetilde{X} \times_X Y \\ \downarrow & & \downarrow \\ X & \longleftarrow & \widetilde{X} \end{array}$$

Pull back the gerbe bundle $L \longrightarrow Y^{[2]}$ and the difference map $\chi : \widetilde{X}^{[2]} \longrightarrow \mathbb{Z}$ to $Y^{[2]} \times_X \widetilde{X}^{[2]}$ and then

$$L^\chi \longrightarrow Y^{[2]} \times_X \widetilde{X}^{[2]}$$

is a bigerbe representing $\gamma \cup \beta \in H^4(X;\mathbb{Z})$ (here $L^\chi = L^{\otimes n}$ on $\chi^{-1}(n)$).

Thank You!

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