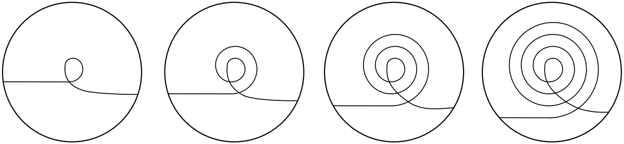
Divide knots of maximal genus defect

CIRGET Séminaire de géométrie et topologie, July 24, 2020



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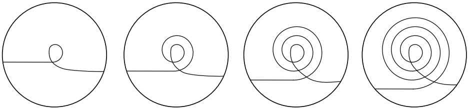
Main results

Theorem (L. '20)

For every positive integer g, there exists a divide knot K_g with smooth four-genus equal to g and with topological four-genus equal to one.

Realised by the examples K_g:



etc.

Corollary

The ratio between the topological and the smooth four-genus can be arbitrarily close to zero for divide knots, and hence for strongly quasipositive fibred knots.

Plan of the talk

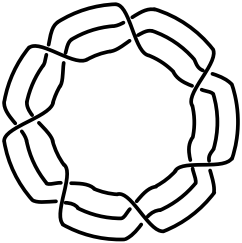
I. Algebraic knots, generalisations and their 4-genera
II. Divide knots

I. Algebraic knots

Let $f(x, y) \in \mathbb{C}[x, y]$ with an isolated singularity at the origin, for example $f(x, y) = x^3 - y^7$.

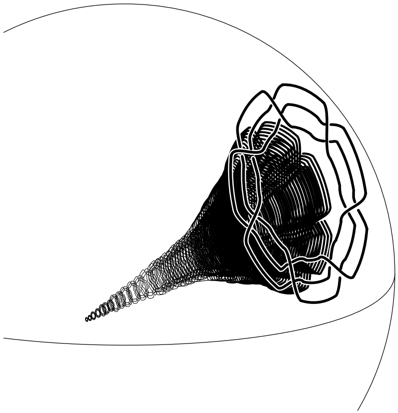
Then $K = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\} \cap S_\varepsilon$ is a link for ε small enough.

For simplicity, we assume it is a knot. In our example, it is the 3,7-torus knot:

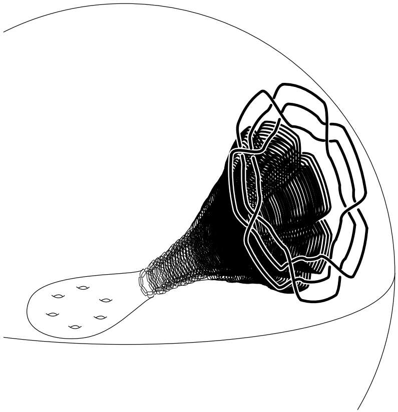


Remark

The zero set $\{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\} \cap B_\varepsilon$ *is a topologically embedded disc and therefore minimises the genus among all surfaces properly embedded in* B_ε *and bounded by* K.

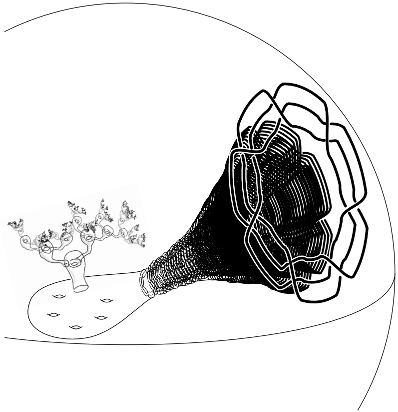


For z regular and small enough, $\{(x, y) \in \mathbb{C}^2 : f(x, y) = z\} \cap B_\varepsilon$ is a smoothly embedded surface and still bounded by K.



By the Thom conjecture (Kronheimer-Mrowka '94): this minimises the genus among all surfaces properly and smoothly embedded in B_ε and bounded by K.

If we require only a locally-flat embedding*, then the genus of a surface bounded by $T_{3,7}$ can be reduced to 5 (using Freedman's disc theorem):



*: locally, the pair (B_ε, surface) is homeomorphic to $(\mathbb{R}^4, \mathbb{R}^2)$.

Picture credit: The iterative part of the figure is copied from Danny Calegari's notes.

4-genera

Let K be a knot in S^3.

Definition

The smooth four-genus $g_4^{\mathrm{smooth}}(K)$ of K is the minimal genus among all surfaces embedded properly, smoothly in the 4-ball and bounded by K.

Definition

The topological four-genus $g_4^{\mathrm{top}}(K)$ of K is the minimal genus among all surfaces embedded properly, locally flatly in the 4-ball and bounded by K.

Definition

The genus defect of K is the difference between the smooth and the topoogical four-genus of K.

For p, q large enough, the genus defect of $T_{p,q}$ also becomes large [Rudolph '84, Lee-Wilczyński '97, Baader-Feller-Lewark-L. '18, McCoy '19].

Theorem (Baader-Banfield-Lewark '20)

$$\limsup_{p,q\to\infty} \frac{g_4^{\mathrm{top}}(T_{p,q})}{g_4^{\mathrm{smooth}}(T_{p,q})} \leq 14/27.$$

Remark

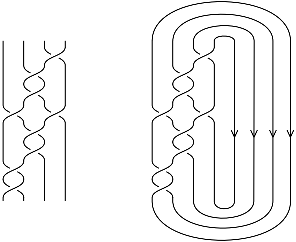
For all coprime p, q:

$$\frac{g_4^{\mathrm{top}}(T_{p,q})}{g_4^{\mathrm{smooth}}(T_{p,q})} \geq 1/2.$$

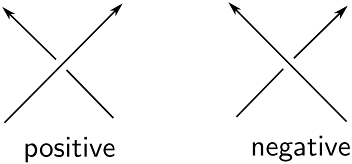
Reason: $\sigma(T_{p,q}) \geq g(T_{p,q}) = g_4^{\mathrm{smooth}}(T_{p,q})$ by Gordon-Litherland-Murasugi '81 and $2g_4^{\mathrm{top}}(K) \geq |\sigma(K)|$ by Kauffman-Taylor '76.

Positive braids and positive knots

Positive braid knots are closures of positive braids:



Positive knots are knots that admit a diagram with only positive crossings:



Genus defect of positive braids and positive knots

Theorem (Feller '18)

For every positive braid knot K,

$$\frac{g_4^{\mathrm{top}}(K)}{g_4^{\mathrm{smooth}}(K)} \geq 1/8.$$

Theorem (Baader-Dehornoy-Liechti '18)

For every positive knot K,

$$\frac{g_4^{\mathrm{top}}(K)}{g_4^{\mathrm{smooth}}(K)} \geq 1/12.$$

Conjecture (Feller)

$$\geq 1/2$$

.

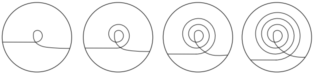
Main results, again

Divide knots are also a generalisation of algebraic knots, but they behave differently with respect to four-dimensional topology.

Theorem (L. '20)

For every positive integer g, there exists a divide knot K_g with smooth four-genus equal to g and with topological four-genus equal to one.

Realised by the examples K_g:



etc.

Corollary

The ratio between the topological and the smooth four-genus can be arbitrarily close to zero for divide knots, and hence for strongly quasipositive fibred knots.

II. Divide knots (A'Campo '98)

Let D be the closed unit disc in $\mathbb{R}^2$.

Definition

A divide P is the image of a relative smooth arc immersed generically in D.

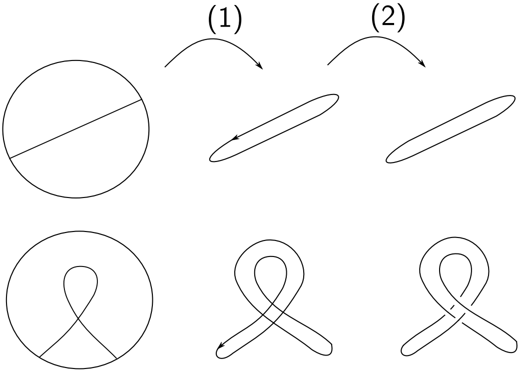
We identify the tangent bundle $T(D)$ to D with the product $D \times \mathbb{R}^2$ and consider its unit sphere

$$ST(D) = \{(x, v) \in T(D) : ||x||^2 + ||v||^2 = 1\} \cong S^3.$$

Definition

The knot $K(P)$ of the divide P is the set of vectors in $ST(D)$ based at and tangent to P.

Examples

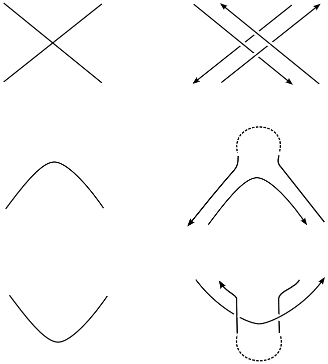


The unknot

The trefoil knot

Algorithm for a diagram

Thinking of ST(D) as a cylinder with identifications, where the height is given by $\arg(v) \in [-\pi, \pi]$ gives the following algorithm (Hirasawa '02):



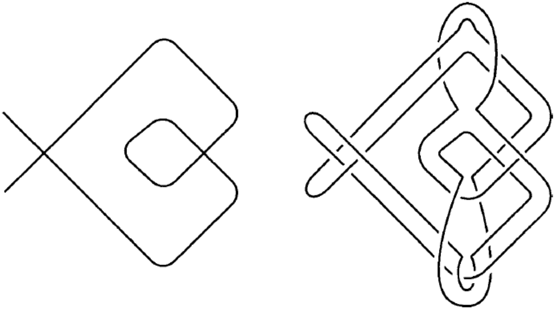
Properties of Divides

Divide knots

- generalise algebraic knots (A'Campo, Gusein-Zade 70s),
- are fibred (A'Campo '98),
- can be drawn algorithmically, together with their fibre surface (Hirasawa '02),
- have double points; their number equals the Seifert genus,
- are strongly quasipositive (Ishikawa '02: explicit Hopf plumbing construction),
- satisfy equality between Seifert genus and smooth 4-genus (Thom conjecture by Kronheimer-Mrowka '94, and Rudolph's extension '93),
- have $|\sigma| \geq 2$ if nontrivial.

$K_2 = 10_{145}$

An isotopy through generically immersed arcs lifts to S^3 and does not change the divide knot. For our second knot K_2, we get:

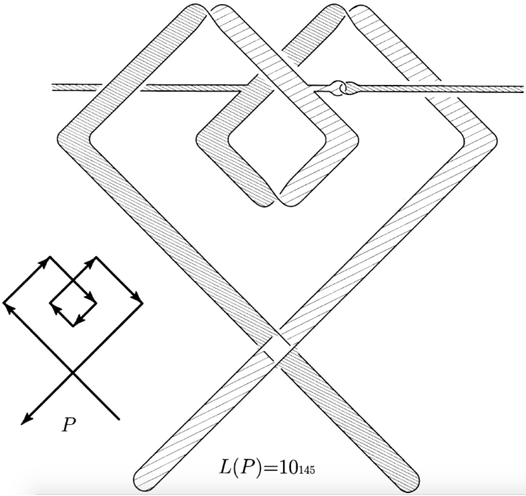


P $\qquad$ $L(P) = 10_{145}$

Picture copied from Hirasawa '02.

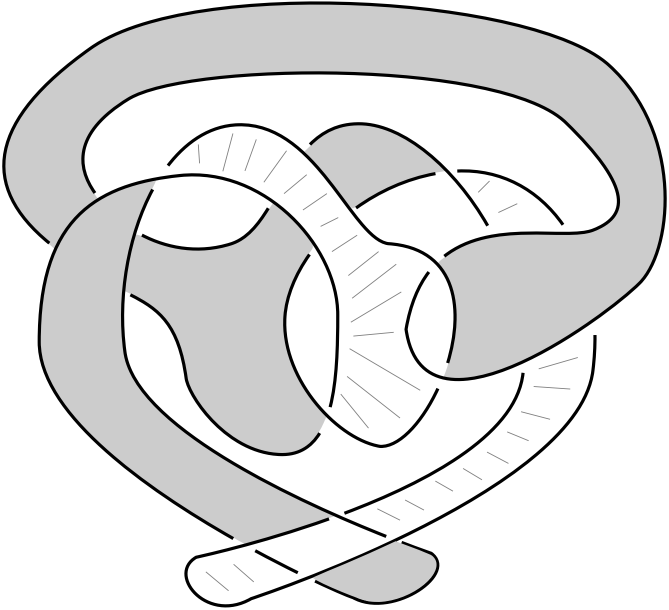
10_{145}

And the fibre surface:

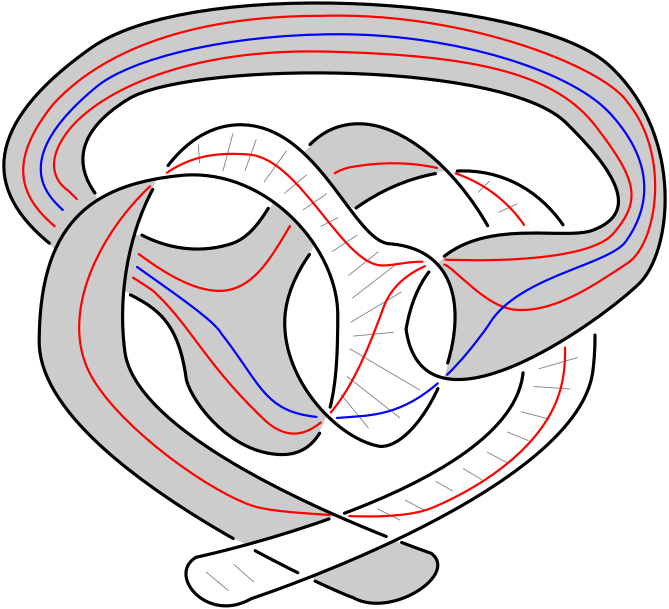


Picture copied from Hirasawa '02.

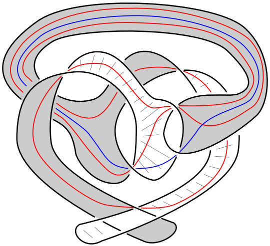
Topological 4-genus of 10_{145}



Topological 4-genus of 10_{145}



Topological 4-genus of 10_{145}



$$\mathrm{lk}(\alpha, \alpha^\sharp) = 0,$$
$$\mathrm{lk}(\beta, \alpha^\sharp) = 0,$$
$$\mathrm{lk}(\alpha, \beta^\sharp) = 1,$$
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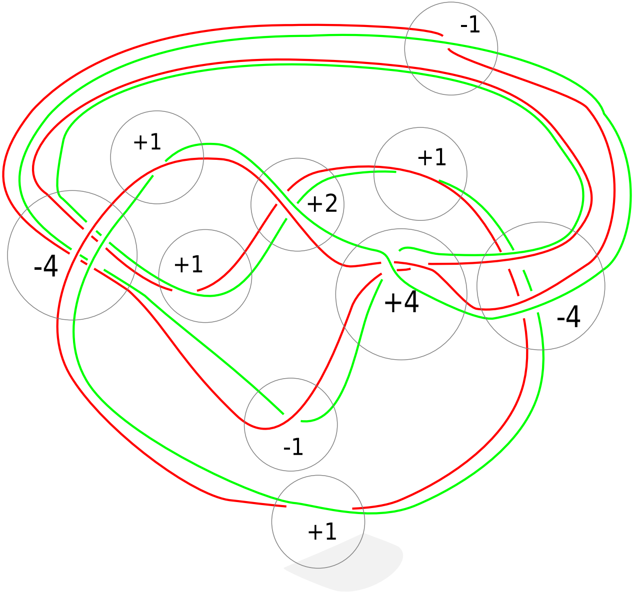
$$\mathrm{lk}(\alpha, \alpha^\sharp) = 0$$



$g_4^{\mathrm{top}}(10_{145}) = 1$

$N = N(\alpha \cup \beta)$ is a Seifert surface for $K = \partial N(\alpha \cup \beta)$. The curves α, β represent a basis of $\mathrm{H}_1(N)$, so a Seifert matrix for K is

$$\begin{pmatrix} \mathrm{lk}(\alpha, \alpha^\sharp) & \mathrm{lk}(\alpha, \beta^\sharp) \\ \mathrm{lk}(\beta, \alpha^\sharp) & \mathrm{lk}(\beta, \beta^\sharp) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

The Alexander polynomial of K is

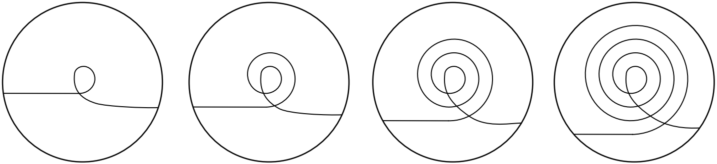
$$\det\left(t^{1/2} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - t^{-1/2} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right) =$$

$$= \det\left(\begin{pmatrix} 0 & t^{1/2} \\ -t^{-1/2} & t^{1/2} - t^{-1/2} \end{pmatrix} \right) = 1.$$

Freedman's disc theorem: K bounds a locally-flat disc D. To obtain a locally-flat genus-one surface embedded in B^4 and bounded by K_2, cut out N and glue in the locally-flat disc D.

Sketch of proof

Consider the sequence K_g:



etc.

We have seen:

- $g_4^{\mathrm{smooth}}(K_g) =$ number of double points $= g$.
- $g_4^{\mathrm{top}}(K_g) \geq |\sigma|/2 \geq 1$.

We need to show $g_4^{\mathrm{top}}(K_g) \leq 1$.

Sketch of proof

Proposition

Let Σ be the fibre surface for K_g. Then $\mathrm{H}_1(\Sigma; \mathbf{Z})$ has a subgroup V of rank $2g-2$ such that for a matrix A of the Seifert form of Σ restricted to V, $\det(t^{1/2}A - t^{-1/2}A^\top) = \pm 1$.

Proposition (Baader-Feller-Lewark-L. '18)

Let L be a link with Seifert surface Σ and associated Seifert form S. If $V \subset \mathrm{H}_1(\Sigma; \mathbf{Z})$ is a subgroup so that for a matrix A of S restricted to V, $\det(t^{1/2}A - t^{-1/2}A^\top) = \pm 1$, then the topological four-genus of L is bounded from above by $g(\Sigma) - \mathrm{rk}(V)/2$.

This finishes the proof of the theorem.

Strongly quasipositive fibred knots

Corollary

The ratio between the topological and the smooth four-genus can be arbitrarily close to zero in for divide knots, and hence for strongly quasipositive fibred knots.

Question

Is there a nontrivial strongly quasipositive fibred knot that is topologically slice?

The question is trivial if we do not ask for fibredness: every Alexander polynomial is realised by a strongly quasipositive knot (Rudolph '83). Even more: every knot is concordant to a strongly quasipositive one (Borodzik-Feller '19).

Merci!