Curvature formula for direct images of relative canonical bundles with a Poincaré type twist

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Topology and Geometry seminar at UQAM

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Direct images of adjoint line bundles

We consider

$$\begin{array}{ccc} L & \longrightarrow & \mathcal{X} \\ & & \downarrow{\scriptstyle f} \\ & & S \end{array}$$

$f : \mathcal{X} \to S$ proper holomorphic submersion of complex manifolds
L line bundle

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Theorem (Berndtsson '09)

If L is (semi-)positive, then the direct image sheaf $f_*(K_{\mathcal{X}/S} + L)$ is locally free and (semi-)positive in the sense of Nakano.

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In case L is relatively positive, there is an intrinsic curvature formula!

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$f : \mathcal{X} \to S$ *a family,* $\mathcal{X}$ *Fano* $\Longrightarrow S$ *Fano*

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Proof.

$L = -K_{\mathcal{X}}$ is positive $\overset{Berndtsson}{\Longrightarrow}$
$f_*(K_{\mathcal{X}/S} - K_{\mathcal{X}}) = f_*(K_{\mathcal{X}} - f^*K_S - K_{\mathcal{X}}) = -K_S$ is positive. □

Singular version

Theorem (Berndtsson-Păun)

In the same setting as above and (L, h) merely pseudoeffective, the direct image sheaf

$$f_*((K_{X/S} + L) \otimes \mathcal{I}(h)) \subset f_*(K_{X/S} + L)$$

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- Closely related question: Positivity properties of $mK_{X/S}$.
- No curvature formula in general that has a geometric interpretation!

Main Motivation

Conjecture (Iitaka)

If $f : X \to Y$ is an algebraic fiber space, i.e. X and Y are non-singular projective and f is surjective with connected fibers, we have

$$\kappa(X) \geq \kappa(Y) + \kappa(X/Y),$$

where $\kappa(X/Y)$ is the Kodaira dimension of a general fiber of f.

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Main ingredients: Positivity properties of $mK_{X/Y}$ and $f_*(mK_{X/Y})$.

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- u is a function on $\mathcal{X}'$ s.t. $u|_{X_s'} \in \mathscr{C}^{k,\alpha}$ for all s and the map $s \mapsto u|_{X_s'}$ is Fréchet differentiable
- $\omega_s := -\mathrm{i}\partial\bar{\partial} \log(h)|_{X_s'}$ is a Poincaré type Kähler metric on each fiber X_s'.

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The curvature form of the hermitian line bundle $(\mathcal{L}, h)$ restricted to $\mathcal{X}'$ is given by

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We say: h^{-1} has **Poincaré type singularities** along $\mathcal{D}$.

Bundle of Logarithmic n-forms

On the open fiber X'_s consider the bundle of L^2 integrable holomorphic n-forms with values in $L_s = \mathcal{L}|_{X_s}$:

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More globally, we have

$$\mathcal{O}_{(2)}\left(K_{\mathcal{X}'/S} \otimes \mathcal{L}|_{\mathcal{X}'}\right) \cong K_{\mathcal{X}/S} \otimes \mathcal{D} \otimes \mathcal{L} = \Omega^n(\log \mathcal{D})_{\mathcal{X}/S}(\mathcal{L})$$

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we have $\psi_s := \psi|_{X'_s} \in H^0(X_s, K_{X_s} \otimes D_s \otimes L_s) = H^0_{(2)}(X'_s, K_{X'_s} \otimes L_s|_{X'_s})$,
thus we define the natural L^2 metric by

$$\langle \psi\, ,\, \phi \rangle_{L^2}(s) = \int_{X'_s} (\psi_s \cdot \overline{\phi_s}) \frac{\omega_s^n}{n!} = \mathrm{i}^{n^2} \int_{X'_s} (\psi_s \wedge \overline{\phi_s}) \cdot h.$$

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$$f_*(v_s) = \partial/\partial s \quad \text{and} \quad \langle v_s, u \rangle_{\omega_{\mathcal{X}'}} = 0 \quad \text{for all } u \in T_{\mathcal{X}'/S}.$$

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It holds

$$(\mathcal{L}, h) > 0 \text{ if and only if } \varphi > 0.$$

Curvature formula

Theorem (N.)

Let $\mathcal{D} \stackrel{i}{\hookrightarrow} \mathcal{X} \stackrel{f}{\to} S$ be a family of smooth log pairs and $(\mathcal{L}, h) \to \mathcal{X}$ a hermitian line bundle as described above. With the objects just described, the L^2-metric on $f_*(K_{\mathcal{X}/S} \otimes \mathcal{D} \otimes \mathcal{L})$ is smooth and its curvature Θ is given by

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$$\langle \Theta \psi, \psi \rangle_{L^2}(s) = \int_{X_s'} \varphi \cdot (\psi \cdot \overline{\psi}) \frac{\omega_s^n}{n!} + \int_{X_s'} (\Box + 1)^{-1}(A \cup \psi) \cdot (\overline{A} \cup \overline{\psi}) \frac{\omega_s^n}{n!}$$

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$$\langle \Theta \psi, \psi \rangle_{L^2}(s) = \int_{X'_s} \varphi \cdot (\psi \cdot \overline{\psi}) \frac{\omega_s^n}{n!} + \int_{X'_s} (\Box + 1)^{-1}(A \cup \psi) \cdot (\overline{A} \cup \overline{\psi}) \frac{\omega_s^n}{n!}$$

In particular $f_*(K_{\mathcal{X}/S} \otimes \mathcal{D} \otimes \mathcal{L})$is Nakano (semi-)positive if $\mathcal{L}$ is (semi-)positive on $\mathcal{X}'$ and positive along the fibers X'_s.

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Theorem (N.)

The curvature of the hermitian metric on $K_{\mathcal{X}'/S}$ that is induced by the Poincaré type Kähler-Einstein metrics on the fibers is semipositive. If the family is effectively parameterized, then $K_{\mathcal{X}'/S}$ is strictly positive.

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For a family of smooth log canonically polarized pairs $\mathcal{D} \overset{i}{\hookrightarrow} \mathcal{X} \overset{f}{\to} S$ *the relative log canonical bundle* $K_{\mathcal{X}/S} \otimes \mathcal{D}$ *equipped with the metric induced from the fiberwise Kähler-Einstein metrics is nef. If the family is effectively parameterized,* $K_{\mathcal{X}/S} \otimes \mathcal{D}$ *is big.*

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By combining both theorems we get

Corollary

For a family of log canonically polarized pairs $\mathcal{D} \stackrel{i}{\hookrightarrow} \mathcal{X} \stackrel{f}{\to} S$ *the direct image sheaf* $f_*((K_{\mathcal{X}/S} \otimes \mathcal{D}) \otimes K_{\mathcal{X}/S})$ *is semipositive in the sense of Nakano. In case the family of log pairs is effectively parameterized this direct image is Nakano positive.*

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Let η be a (n, n)-form on $\mathcal{X}'$. Then we have

$$\frac{\partial}{\partial s}\int_{X'_s}\eta = \int_{X'_s} L_v(\eta),$$

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Lemma

$$\frac{\partial}{\partial s}\langle\psi,\psi\rangle = \langle L_v\psi,\psi\rangle + \langle\psi, L_{\overline{v}}\psi\rangle,$$

where $L_v = \delta_v\nabla + \nabla\delta_v$ using the hermitian connection ∇ on $\mathcal{L}$-valued $(n, 0)$-forms on $\mathcal{X}'$.

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Define another metric on A by

$$h_{A,\varepsilon} := h_A \cdot \left(\prod_i |\sigma_i|^2_{h_i} \log^2(|\sigma_i|^2_{h_i}) \right)^{\varepsilon},$$

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$$\mathrm{i}\,\Theta_{h_{A,\varepsilon}}(A) = \mathrm{i}\,\Theta_{h_A}(A) - \varepsilon\sqrt{-1}\sum_i \partial\bar{\partial}\log\left(|\sigma_i|^2_{h_i}\log^2(|\sigma_i|^2_{h_i})\right) > 0 \quad \text{on} \quad \mathcal{X}'$$

for some $\varepsilon > 0$.

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$$f_*((K_{\mathcal{X}/S} + F) \otimes \mathcal{J}(h_F)) = f_*(K_{\mathcal{X}/S} + F + \mathcal{O}(-E)) = f_*(K_{\mathcal{X}/S} + A)$$

is positive.

End of talk

Thanks for your attention!