The Hitchin-cscK system
an infinite-dimensional hyperkähler reduction

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General setting

cscK problem

(X^n, ω_0) compact Kähler manifold, we look for a Kähler form $\omega \in [\omega_0]$ with constant scalar curvature

$$S(\omega) = \widehat{S(\omega)}.$$

- fourth order fully nonlinear PDE;

- Fujiki, Donaldson 1990s: this is a moment map equation for an infinite-dimensional Hamiltonian action $\mathcal{G} \curvearrowright \mathcal{J}$

There *should* be an algebraic notion of stability characterizing existence of solutions. This is motivated by the Kempf-Ness Theorem, relating symplectic quotients with GIT quotients.

Moment map picture

Fix a Kähler form ω_0 on X, consider

$$\mathcal{J} = \text{almost complex structures compatible with } \omega_0$$

$\mathcal{J}$ is a Kähler manifold; for $J \in \mathcal{J}$, consider first-order deformations

$$T_J\mathcal{J} = \left\{ \alpha : T^{1,0}_J X \to T^{0,1}_J X \;\middle|\; \omega_0(\alpha -, -) + \omega_0(-, \alpha -) = 0 \right\}$$

the Kähler metric is $\langle \alpha, \beta \rangle = \operatorname{Re} \int_X \operatorname{Tr}(\alpha\bar{\beta}) \frac{\omega_0^n}{n!}$.

The group $\mathcal{G}$ of Hamiltonian symplectomorphisms of (X, ω_0) acts by pull-back on $\mathcal{J}$, and the action is Hamiltonian. The moment map is

$$J \mapsto 2\left(S(J, \omega_0) - \hat{S}\right)$$

Upgrading to a hyperkähler reduction

Our goal is to upgrade the Kähler reduction $\mathcal{G} \curvearrowright \mathcal{J}$ to a *hyper*-Kähler reduction $\mathcal{G} \curvearrowright T^*\mathcal{J}$.

More precisely, we want a hyperkähler structure on $T^*\mathcal{J}$ such that

- the zero-section $\mathcal{J} \hookrightarrow T^*\mathcal{J}$ is a Kähler embedding;
- the induced action $\mathcal{G} \curvearrowright T^*\mathcal{J}$ is Hamiltonian with respect to *all* the Kähler forms in the hyperkähler family.

So we will find three Kähler forms Ω_1, Ω_2 and Ω_3 on $T^*\mathcal{J}$ such that:

- Ω_1 restricted to $\mathcal{J}$ coincides with the Donaldson-Fujiki symplectic structure;
- $\mathcal{G} \curvearrowright (T^*\mathcal{J}, \Omega_i)$ is Hamiltonian, for $i = 1, 2, 3$.

For $\dim X = 1$, the problem has been studied by Donaldson in 2003.

Motivation: Hitchin's Higgs bundles equations

Much of the study of the cscK problem has been inspired by the theory of the Hermitian Yang-Mills equation, particularly in its connection with algebraic stability notions.

The HYM equation

Let (X, ω) be a Kähler curve, $E \to X$ a holomorphic vector bundle. We look for a Hermitian metric h on E such that

$$F(h) = \lambda\, \omega \otimes \mathbb{1}.$$

- moment map equation for an infinite-dimensional Kähler reduction (Atiyah-Bott, ′83);
- group of unitary gauge transformations acting on the infinite dimensional space $\mathcal{A}(E) :=$ $\bar{\partial}$-operators on E;
- existence of solutions equivalent to slope-stability of E.

Motivation: Hitchin's Higgs bundles equations

Hitchin studied a system of equations that is a "hyperkähler extension" of the HYM equation. First we define a hyperkähler structure on $T^*\mathcal{A}(E) = \mathcal{A}(E) \times \mathcal{A}^{1,0}(\mathrm{End}(E))$ that extends the Kähler structure of $\mathcal{A}(E)$.
Then one checks that that the induced action on $T^*\mathcal{A}(E)$ is Hamiltonian with respect to the Kähler forms Ω_1, Ω_2, Ω_3 generating the hyperkähler family.

The moment map equations for $\Theta := \Omega_2 + \mathrm{i}\Omega_3$ and Ω_1 are, for a Hermitian metric h and a Higgs field $\phi \in \mathcal{A}^{1,0}(\mathrm{End}(E))$

$$\begin{cases} \bar{\partial}\phi = 0 \\ F(h) + [\phi, \phi^{*_h}] = \lambda\,\omega \otimes \mathbb{1} \end{cases}$$

Ω_1 extends the Kähler form of $\mathcal{A}$; Θ is the canonical 2-form on $T^*\mathcal{A}$.

Some features of the two problems

HYM equation	**cscK equation**
moment map eqs for an infinite-dimensional Hamiltonian action	
bounds on the Yang-Mills functional	Donaldson's lower bounds on the Calabi functional
equivalent to slope-stability	(conjecturally) equivalent to K-stability

⇓ *hyperkähler extension of the moment map picture* ⇓

Higgs bundle equations	**?**
stability of Higgs bundles	?
moduli space of solutions with rich algebraic and geometric structures	?

Defining a hyperkähler structure

Recall that $\mathcal{J}$ is the set of complex structures compatible with a fixed Kähler form ω_0 on X^n. We can also describe elements of $\mathcal{J}$ as sections of a fibre bundle $\mathcal{F} \to X$ such that

- fibres isomorphic to $\mathrm{Sp}(2n)/U(n)$ (Siegel upper half space)
- $\mathcal{F}$ becomes trivial over any system of Darboux coordinates for ω_0
- the transition function between two trivializations acts as a $\mathrm{Sp}(2n)$-translation on $\mathrm{Sp}(2n)/U(n)$

Biquard and Gauduchon ($'97$) construct a hyperkähler structure on $T^*(G/H)$, for *any* Hermitian symmetric space G/H.
This hyperkähler structure extends the metric on G/H and is invariant under the G-action.

We can use the Biquard-Gauduchon metric on $T^*\left(\mathrm{Sp}(2n)/U(n)\right)$ to define a hyperkähler metric on $T^*\mathcal{J}$.

Lift of the Hamiltonian action to $T^*\mathcal{J}$

We have three symplectic forms Ω_i on $T^*\mathcal{J}$. The form $\Theta = \Omega_2 + \mathrm{i}\Omega_3$ is the canonical holomorphic-symplectic form on $T^*\mathcal{J}$, while Ω_1 extends the Kähler form of $\mathcal{J}$.
The Hamiltonian action $\mathcal{G} \curvearrowright \mathcal{J}$ lifts to an action on $T^*\mathcal{J}$ that is Hamiltonian with respect to both Θ and Ω_1.

we get a system on X of moment map equations for a Kähler form ω and a first-order deformation of the complex structure $\alpha \in \mathcal{A}^{1,0}(T^{0,1}X)$

Hitchin-cscK system
$$\begin{cases} \operatorname{div}\left(\partial^*\alpha\right) = 0 \\ 2\left(S(\omega) - \hat{S}\right) + \operatorname{div}\left(X(\omega,\alpha)\right) = 0 \end{cases}$$

We will see later the (slightly cumbersome) expression of the vector field $X(\omega, \alpha)$.

Some remarks

- The Biquard-Gauduchon hyperkähler metric on $T^*\left(\mathrm{Sp}(2n)/U(n)\right)$ is defined only on an open neighbourhood of the zero section
- the metric on this neighbourhood is not complete

Our hyperkähler metric is defined only on an open submanifold $\mathcal{U} \subset T^*\mathcal{J}$;

$$(J,\alpha) \in \mathcal{U} \iff \text{the eigenvalues of } \alpha\bar{\alpha} \text{ are less than } 1$$

So we should add two more equations to the HcscK system: $r(\alpha\bar{\alpha}) < 1$ and $\omega(-,\alpha-) \in S^2(T^*X)$.

Now, let $\hat{\alpha} = \left(1+\sqrt{1-\alpha\bar{\alpha}}\right)^{-1}$ and $g := \omega(-,J-)$. Then

$$X(\omega,\alpha) = \mathrm{Re}\left(g(\nabla^a\alpha,\bar{\alpha}\hat{\alpha})\partial_{z^a} - g(\nabla^{\bar{b}}\alpha,\bar{\alpha}\hat{\alpha})\partial_{\bar{z}^b} - 2\nabla^*(\alpha\bar{\alpha}\hat{\alpha})\right)$$

Some results

The system is somewhat easier to study in dimension 1 and 2, as they depend on the matrix square root $\sqrt{1-\alpha\bar{\alpha}}$.

In dimension 1 the construction was carried out by Donaldson in 2003, and a special set of solutions has been studied by Hodge.

Theorem (Hodge '05, S.-Stoppa '20)

Solutions α of the equation $\operatorname{div}(\partial^*\alpha) = 0$ *are parametrized by pairs* $(\beta, \tau) \in H^0(K_X) \times H^0(K_X^2)$*. On a high-genus curve, for every small* (β, τ) *there is a unique Kähler form* ω *solving the HcscK system.*

Some results

The 2-dimensional case is the first in which the cscK problem is obstructed. Consider in particular a high-genus curve $\mathcal{C}$ and let $X = \mathbb{P}(\mathcal{O} \oplus K_{\mathcal{C}})$. There is no cscK metric on X, as the surface is unstable. However, we can stabilize the surface by adding a suitable "Higgs term" to the cscK problem.

Theorem (S.-Stoppa '18)

Let ω_0 be the hyperbolic metric on $\mathcal{C}$ and let $\mathcal{O}(1)$ be the relative hyperplane bundle of $X \to \mathcal{C}$. For all sufficiently small $m > 0$, there is a deformation of the complex structure α and a Kähler metric ω in

$$[\omega_m] = [\pi^* \omega_{\mathcal{C}}] + m\, c_1(\mathcal{O}(1))$$

such that (ω, α) is a solution of the HcscK system.

Symplectic coordinates

There is a situation in which the HcscK system has a much simpler expression. Assume that there is a simply connected open dense subset in X over which we have complex coordinates $\boldsymbol{z} = \boldsymbol{x} + \mathrm{i}\boldsymbol{w}$.

We consider Kähler forms ω that are invariant under translations in the $\boldsymbol{w}$-variables, so the matrix associated to the Kähler metric in the coordinates $\boldsymbol{z}$ is the Hessian of a convex function $v(\boldsymbol{x})$. Consider the Legendre dual $u(\boldsymbol{y})$ of $v(\boldsymbol{x})$. In this new coordinates, the scalar curvature of ω has a particularly simple expression:

Theorem (Abreu '98)

Let u_{ij} be the Hessian of u, and u^{ij} its inverse. Then

$$S(\omega) = -\frac{1}{4} u^{ij}_{,ij}.$$

The HcscK system in symplectic coordinates

Let now α be a deformation of the complex structure, invariant under $\boldsymbol{w}$-translations. Under Legendre duality, α is described in terms of a symmetric complex matrix $\xi(\boldsymbol{y}) \in \mathcal{C}^\infty(\mathbb{C}^{n\times n})$.

The HcscK system written in terms of the symplectic potential u and the deformation ξ becomes ($G := \mathrm{Hess}(u)$)

$$\begin{cases} \xi^{ij}_{,ij} = 0 \\ \left(\sqrt{1 - \xi G \bar{\xi} G}\; G^{-1}\right)^{ij}_{,ij} = -4\hat{S} \end{cases} + \text{boundary conditions for } \xi \text{ and } u.$$

- abelian varieties: $\mathbb{Z}^n$-periodicity of $u_{ij}(\boldsymbol{y})$ and $\xi^{ij}(\boldsymbol{y})$;
- toric varieties: Guillemin's boundary conditions for $u(\boldsymbol{y})$. ξ?

Abelian varieties

On an abelian surface $X \cong \mathbb{C}^2/\mathbb{Z}^2 + \mathrm{i}\mathbb{Z}^2$ the problem is unobstructed. If we fix a sufficiently small ξ such that $\xi^{ij}_{,ij} = 0$, then there is a symplectic potential u solving the real moment map equation. Moreover, there is a variational characterization of the real moment map equation, giving a uniqueness result for solutions of the real moment map equation.

Theorem (S.-Stoppa '20)

Let $\mathcal{A}(\xi) = \left\{ u \,\middle|\, \mathrm{Tr}\left(\xi\, G_u\, \bar{\xi}\, G_u\right) < 1 \right\}$ *where* $G_u = \mathrm{Hess}(u)$. *For* $u \in \mathcal{A}(\xi)$, *denote by* $\delta_i(u)$ *the eigenvalues of* $\xi\, G_u\, \bar{\xi}\, G_u$. *The Euler-Lagrange equation of the convex functional*

$$\mathrm{HK}(u) = -\int \log\det(G)\,\mathrm{d}\mu + \int \sum_i \left(\log \frac{1 + \sqrt{1 - \delta_i(u)}}{2} - \sqrt{1 - \delta_i(u)} \right) \mathrm{d}\mu$$

is the real moment map equation of the HcscK system.

Toric varieties: obstructions to the HcscK system

On a toric variety the coordinates $\boldsymbol{x}$ are defined on a convex polytope $P \subset \mathbb{R}^n$. The boundary conditions for u are *Guillemin's boundary conditions*: G^{-1} acquires a kernel along ∂P generated by the vector normal to ∂P.

The boundary conditions for the Higgs term ξ can be expressed in terms of those of u, as ξ can always be written as $\xi = G^{-1}\Phi G^{-1}$ for a smooth symmetric matrix $\Phi \in \mathcal{C}^\infty(\bar{P}, \mathbb{C}^{n\times n})$.

On a toric surface, there are some obstruction to solving the HcscK system: for example, the Higgs term α must *not* be an integrable deformation of the complex structure.

Toric varieties: obstructions to the HcscK system

More interestingly, if the Futaki invariant of the toric surface does not vanish then the real moment map equation can not be solved.

Lemma

There is a measure $\mathrm{d}\sigma$ *on* ∂P *such that, for every smooth* f

$$\int_P \left(\sqrt{1-\xi G\bar{\xi}G}\;G^{-1}\right)^{ij}_{,ij} f\,\mathrm{d}\mu = \int_P \left(\sqrt{1-\xi G\bar{\xi}G}\;G^{-1}\right)^{ij} f_{,ij}\,\mathrm{d}\mu - \int_{\partial P} f\,\mathrm{d}\sigma.$$

So, if (ξ, u) *is a solution to the HcscK system, for every affine-linear* f

$$\int_P 4\hat{S}\,f\,\mathrm{d}\mu - \int_{\partial P} f\,\mathrm{d}\sigma = 0. \qquad (\star)$$

The left hand side of $(\star)$ is the Futaki invariant of the vector field defined by the linear function f (Donaldson, $'02$).

Loose ends and future developments

- Hyperkähler quotient structure?
- Can all of this be tied to some stability notion?
- Is there a similar "Hitchin construction" for other problems in differential geometry?

Thank you!