Families of singular Kähler-Einstein metrics

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Yau's theorem

(X, ω_0) compact Kähler manifold such that $c_1(X) = 0$.

Theorem (Yau '78)

There exists a unique smooth Kähler form $\omega \in [\omega_0]$ such that

$$\mathrm{Ric}\,\omega = 0.$$

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Application : Beauville-Bogomolov decomposition Theorem

There exists $f : X' \to X$ finite unramified cover such that

$$X' = T \times \prod_i Y_i \times \prod_j Z_j$$

where T is a torus, Y_i a Calabi-Yau manifold and Z_j a Hyperkähler manifold.

Generalizations

Some among many generalizations

1. (X, ω) complete non-compact (*Tian-Yau '87*)
2. ω with conic singularities (*Brendle, Campana-G.-Păun, Jeffres-Mazzeo-Rubinstein* $\sim$ *'11*)
3. (X, ω) compact Hermitian (*Tosatti-Weinkove '10*)
4. X mildly singular compact Kähler space (*Eyssidieux-Guedj-Zeriahi, Tian-Zhu, Tsuji - early '00s*)

The EGZ theorem

Definition : Singular Calabi-Yau varieties

X normal compact Kähler space with canonical singularities and $K_X \sim \mathcal{O}_X$ ($\Leftrightarrow \exists\, \Omega$ holomorphic never vanishing n-form on X_{reg} s.t. $i^{n^2} \int_{X_{\mathrm{reg}}} \Omega \wedge \overline{\Omega} < +\infty$).

Theorem (Eyssidieux-Guedj-Zeriahi '06)

X singular CY, ω_0 Kähler metric. There exists a unique Kähler form $\omega \in [\omega_0]|_{X_{\mathrm{reg}}}$ on X_{reg} s.t.

1. $\mathrm{Ric}\,\omega = 0$.
2. $\omega = \omega_0 + dd^c\varphi$ with $\varphi \in L^\infty(X_{\mathrm{reg}})$

(1) $\iff \omega^n = c_0 \cdot \Omega \wedge \overline{\Omega}$
(2) $\iff \int_{X_{\mathrm{reg}}} \omega^n = \int_X \omega_0^n$

Some recent developments

Main challenge

Understand the geometry of the *non-complete*, Kähler Ricci-flat manifold (X_{reg}, ω).

Some recent developments

Recent results and applications

1. Identification of metric completion $\overline{(X_{\text{reg}}, \omega)} \simeq_{\text{top.}} X$ when X admits a projective smoothing (*Rong-Ruan-Zhang '11, Donaldson-Sun '13, Song '14*)
2. Asymptotics of ω near X_{sing} when X has smoothable conical singularities (*Hein-Sun '16*)
3. Computation of the holonomy of (X_{reg}, ω) (*Greb-G.-Kebekus '17*) $\rightsquigarrow$ BB decomposition theorem for singular projective CY (*Druel '16, Höring-Peternell '17*)

Reduction to a Monge-Ampère equation

(X, ω_0) compact Kähler manifold.

Yau theorem $\iff$ solving an equation

$$(\theta + dd^c\varphi)^n = f\omega_0^n$$

where

1. θ is a given Kähler form.
2. $f \in \mathcal{C}^\infty(X, \mathbb{R}_{>0})$ is given, satisfying $\int_X f\omega_0^n = \int_X \theta^n$.
3. $\varphi \in \mathrm{PSH}(X, \theta) \cap \mathcal{C}^\infty(X, \mathbb{R})$ is the unknown.

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$\rightsquigarrow$ $\|\varphi\|_{\mathcal{C}^{k,\alpha}(X)}$ depends on $X, \theta, \|f^{\pm 1}\|_{\mathcal{C}^{k-2,\alpha'}(X)}$.

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1. θ is a given semipositive form, definite somewhere.
2. $f \in L^p(X, \mathbb{R}_{\geq 0})$ is given with $p > 1$ and $\int_X f\omega_0^n = \int_X \theta^n$.
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$\rightsquigarrow$ $\|\varphi\|_{L^\infty(X)}$ depends on $X, \theta, p, \|f\|_{L^p}$. [cf *Kołodziej '98* for θ Kähler]

The dependence in X

- X compact Kähler manifold of dimension n.
- θ semipositive form with $V := \int_X \theta^n > 0$.
- μ proba measure and $f \geq 0$ s.t. $\int_X f d\mu = 1$.

Assumption

There exist constants $p > 1, \alpha > 0, A_\alpha, C_p$ such that

1. $(\int_X f^p d\mu)^{1/p} \leq C_p$.
2. $\int_X e^{-\alpha\psi} d\mu \leq A_\alpha$ for all $\psi \in \mathrm{PSH}(X, \theta)$ with $\sup_X \psi = 0$.

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Theorem (Di Nezza-Guedj-G. '20)

There exists an explicit $M = M(n, p, C_p, \alpha, A_\alpha)$ such the solution φ of the MA equation

$$\frac{1}{V}(\theta + dd^c\varphi)^n = f\,\mu$$

satisfies

$$\sup_X \varphi - \inf_X \varphi \leq M$$

The smooth case

- $p : \mathfrak{X} \to \mathbb{D}$ a proper holomorphic submersion of rel. dim. n
- θ a Kähler form on $\mathfrak{X}$
- Ω a trivialization of $K_{\mathfrak{X}/\mathbb{D}}$ (a non-vanishing rel. n-form)

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Application of the implicit function theorem

The Kähler-Ricci flat metric $\omega_t \in [\theta_t]$ solving

$$\omega_t^n = c_t \, \Omega_t \wedge \overline{\Omega_t}$$

varies smoothly with $t \in \mathbb{D}$.
[c_t is a normalizing constant which varies smoothly with t.]

The singular case

- $(\mathfrak{X}, \theta)$ normal Kähler space
- $p : \mathfrak{X} \to \mathbb{D}$ proper holomorphic surjective of rel. dim. n
- Ω trivialization of $K_{\mathfrak{X}/\mathbb{D}}$ s.t. $i^{n^2} \int_{X_0^{\mathrm{reg}}} \Omega_0 \wedge \overline{\Omega_0} < +\infty$.

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Theorem (Di Nezza-Guedj-G. '20)

$\exists\, \varepsilon_0, M > 0$ s.t. the singular KE metric $\omega_t = \theta_t + dd^c \varphi_t$ solving

$$\omega_t^n = c_t\, \Omega_t \wedge \overline{\Omega_t}$$

satisfies

$$\forall |t| < \epsilon_0, \quad \sup_{X_t} \varphi_t - \inf_{X_t} \varphi_t \leq M.$$

Elements of the proof

1. Control of the L^p norm of the densities $\Omega_t \wedge \overline{\Omega_t}$ wrt θ_t^n.
   $\rightsquigarrow$ local computations in semistable model.
2. Find $\alpha > 0$ such that $\int_{X_t} e^{-\alpha\psi_t}\theta_t^n < +\infty$ for any $\psi_t \in \mathrm{PSH}(X_t, \theta_t)$.
   $\rightsquigarrow$ amounts to controlling Lelong numbers and in turn mass of positive currents in $[\theta_t]$.
3. Control the integrals $\int_{X_t} e^{-\alpha\psi_t}\theta_t^n$ for any $\psi_t \in \mathrm{PSH}(X_t, \theta_t)$ with $\sup_{X_t} \psi_t = 0$.
   $\rightsquigarrow$ Work with local pluricomplex Green functions to reduce to a control of mass and...
4. Show that $\psi_t \in \mathrm{PSH}(X_t, \theta_t)$ with $\sup_{X_t} \psi_t = 0$ implies $\int_{X_t} \psi_t \theta_t^n \geq -C$.

Generalizations

Other contexts

1. Families of manifolds of general type
2. KE metrics with conic singularities
3. Negative curvature: families of stable varieties