Sasaki-Einstein Metrics and the Iterated Join

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Kähler-Einstein metrics:

- N is a smooth compact mani(orbi-)fold of real dimension $2d_N$.
- Kähler structure (N, J, g, ω)
- $[\omega] \in H^2(N, \mathbb{R})$ is called the **Kähler class**.
- (For fixed J, the subset in $H^2(N, \mathbb{R})$ consisting of Kähler classes is called the **Kähler cone**.)
- the Riemannian **Ricci tensor** $r : TN \otimes TN \to C^\infty(N)$ gives us the **Ricci form**, $\rho(X, Y) = r(JX, Y)$.
- The miracle of Kähler geometry is that $c_1(N, J) = [\frac{\rho}{2\pi}]$.
- If $\rho = \lambda\omega$, where λ is some constant, then we say that (N, J, g, ω) is **Kähler-Einstein** (or just **KE**).
- KE $\implies$ $c_1(N, J)$ is positive, negative, or null.
- & KE $\implies$ Constant Scalar Curvature, **CSC** (with $\lambda = \frac{Scal}{2d_N}$)

Sasakian geometry:

the odd dimensional sibling of Kählerian geometry.

A Sasakian structure on a smooth compact manifold M of dimension $2n+1$ is defined by a quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ where

- η is **contact 1-form** defining a subbundle (contact bundle) in TM by $\mathcal{D} = \ker \eta$.
- ξ is the **Reeb vector field** of η $[\eta(\xi) = 1$ and $\xi \rfloor d\eta = 0]$
- Φ is an endomorphism field which annihilates ξ and satisfies $J = \Phi|_{\mathcal{D}}$ is a complex structure on the contact bundle $(d\eta(J\cdot, J\cdot) = d\eta(\cdot,\cdot))$
- $g := d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$ is a Riemannian metric
- ξ is a Killing vector field of g which generates a one dimensional foliation $\mathcal{F}_\xi$ of M whose transverse structure is Kähler.
- $(dt^2 + t^2 g, d(t^2\eta))$ is Kähler on $M \times \mathbb{R}^+$ with complex structure I: $IY = \Phi Y + \eta(Y) t\frac{\partial}{\partial t}$ for vector fields Y on M, and $I(t\frac{\partial}{\partial t}) = -\xi$.

Corresponding Kähler Foliations

- If ξ is **regular**, M has a free S^1 action generated by ξ and the quotient of the foliation $\mathcal{F}_\xi$ is compact Kähler manifold N whose Kähler cohomology class lies in $H^2(N,\mathbb{Z})$.
- If ξ is **quasi-regular**, M has a locally free S^1 action generated by ξ and the quotient of the quasi-regular foliation is a compact Kähler orbifold $\mathcal{Z}$ with finite cyclic local uniformizing groups whose orbifold Kähler cohomology class lies in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$.
- If not regular or quasi-regular, we call it **irregular**... (that's most of them...but we shall mostly ignore them here)
- In the quasi-regular case, the orbifold Chern class $c_1^{orb}(\mathcal{Z})$ pulls back to the basic first Chern class $c_1(\mathcal{F}_\xi)$.

Transverse Homothety:

- If $\mathcal{S}=(\xi,\eta,\Phi,g)$ is a Sasakian structure, so is $\mathcal{S}_a=(a^{-1}\xi,a\eta,\Phi,g_a)$ for every $a\in\mathbb{R}^+$ with $g_a=ag+(a^2-a)\eta\otimes\eta$.
- So Sasakian structures come in rays.

Deforming the Sasaki structure:

In its contact structure isotopy class:

- $\eta \to \eta + d^c\phi, \quad \phi \quad$ is basic
- ϕ is basic: $\xi \rfloor \phi = 0$ and $\mathcal{L}_\xi \phi = 0$
- This corresponds to a deformation of the transverse Kähler form $\omega_T \to \omega_T + dd^c\phi$ in its basic Kähler class (or simply induced Kähler class on the orbifold in the regular/quasi-regular case).
- "Up to isotopy" means that the Sasaki structure might have been deformed as above.

In the Sasaki Cone:

- Pick maximal T^k, $0 \leq k \leq n+1$ in the Sasaki automorphism group $\mathfrak{Aut}(\mathcal{S}) = \{\phi \in \mathcal{D}iff(M) \,|\, \phi^*\eta = \eta, \phi^*J = J, \phi^*\xi = \xi, \phi^*g = g\}$.
- The unreduced Sasaki cone is $\mathfrak{t}^+ = \{\xi' \in \mathfrak{t}_k \,|\, \eta(\xi') > 0\}$
- Each element in $\mathfrak{t}^+$ determines a new Sasaki structure with the same underlying CR-structure.

Sasaki-Einstein metrics

- The Ricci tensor of g behaves as follows:
  - $r(X,\xi) = 2n\,\eta(X)$ for any vector field X
  - $r(X,Y) = r_T(X,Y) - 2g(X,Y)$, where X, Y are sections of $\mathcal{D}$ and r_T is the transverse Ricci tensor
- The scalar curvature of g behaves as follows $Scal = Scal_T - 2n$
- If the transverse Kähler structure is Kähler-Einstein then we say that the Sasaki metric is η-Einstein.
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ is η-Einstein iff its entire ray is η-Einstein
- If η-Einstein and $Scal_T > 0$, then exactly one of the Sasaki structures in the η-Einstein ray is actually Sasaki-Einstein.
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is Sasaki-Einstein, then we must have that $c_1(\mathcal{D})$ is a torsion class (for example, it vanishes).
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ has constant scalar curvature (CSC) iff the transverse Kähler structure has constant scalar curvature.
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ has CSC iff its entire ray has CSC ("CSC ray").

S^1-orbibundles (briefly)

Assume that N is a normal (compact) projective algebraic variety with a fixed orbifold structure which we write as the pair (N, Δ_N) where Δ_N is a sum of irreducible branch divisors:

$$\Delta_N = \sum_{j=1}^{k} (1 - \frac{1}{m_j}) D_j$$

where $k = \dim \mathrm{Div}(N)$, the group of Weil divisors on N, m_j are the ramification indices, and $D_j \in \mathrm{Div}(N)$. The algebraic singular locus is also given an orbifold structure. They are finite cyclic quotient singularities. [see Chapt. 4 in book by Boyer and Galicki]

By Theorem 4.3.15 in Boyer and Galicki's book the isomorphism classes of S^1 orbibundles, M, over (N, Δ_N) are uniquely determined by their orbifold first Chern class denoted by $c_1^{orb}(M/N) \in H^2(N, \mathbb{Q})$. Furthermore, when M has a Sasakian structure, $c_1^{orb}(M/N)$ is an orbifold Kähler class with Kähler form denoted by ω_N.

Generalized orbifold Calabi construction

History:

The generalized Calabi construction was presented by Apostolov, Calderbank, Gauduchon, and T-F. Here we make a modest generalization, allowing for the base space to be a normal projective algebraic variety with cyclic orbifold singularities. Other than that, it is staying in the "admissible" niche of the construction. In this form, credit is due to Calabi, Guan, Hwang, Koiso, LeBrun, Pedersen, Poon, Sakane, Singer, Simanca, and others.

The ingredients in the construction are as follows:

- [Base] A cyclic Kähler orbifold (N, Δ_N) equipped with a Kähler orbifold metric, g_{base}, whose Kähler form, $\omega_{base} = 2\pi\omega_N$, satisfies that $[\omega_N]$ is a primitive orbifold Kähler class. Note that a primitive orbifold class $[\omega_N]$ is obtained from a primitive integer class $[\omega_N]_I$, viz $[\omega_N] = \frac{[\omega_N]_I}{\Upsilon_N}$. ($\Upsilon_N$ is order of (N, Δ_N))
- [Fiber] A weighted projective line $(\mathbb{CP}^1_{\mathbf{m}} = \mathbb{CP}^1_{v^0,v^\infty}/\mathbb{Z}_m, g_{\mathbf{m}}, \omega_{\mathbf{m}})$ with orbifold Kähler structure $(g_{\mathbf{m}}, \omega_{\mathbf{m}})$. Here $(m^0, m^\infty) = m(v^0, v^\infty)$ and v^0, v^∞ are coprime.
- A principal S^1 orbi-bundle, $P_n \to (N, \Delta_N)$, with a principal connection of curvature $n\,\omega_{base} \in \Omega^{1,1}((N, \Delta_N), \mathbb{R})$, where S^1 acts on $\mathbb{CP}^1_{\mathbf{m}}$, $n \in \mathbb{Z} \setminus \{0\}$, and $\gcd(n, m) = 1$.
- A constant $0 < |r| < 1$ with the same sign as n

From this data we may define the orbifold

$$(S_n, \Delta_{\mathbf{m},N}) = P_n \times_{S^1} \mathbb{CP}^1_{\mathbf{m}} \to (N, \Delta_N)$$

$\Delta_{\mathbf{m},N} = \Delta_{\mathbf{m}} + \pi^{-1}(\Delta_N) = (1 - \frac{1}{m^0})D^0 + (1 - \frac{1}{m^\infty})D^\infty + \sum_{j=1}^k (1 - \frac{1}{m_j})\pi^{-1}(D_j)$
Where $\Delta_{\mathbf{m}}$ is a branch divisor consisting of the zero D^0 and infinity D^∞ sections of $S_n = \mathbb{P}(\mathbb{1} \oplus L_n)$ with ramification indices
$\mathbf{m} = (m^0, m^\infty) = m(v^0, v^\infty)$.Moreover, $c_1^{orb}(L_n) = n[\omega_N]$

We have the commutative diagram

$$\begin{array}{ccc} (S_n, \Delta_{\mathbf{m},N}) & \xrightarrow{\pi^{orb}} & (N, \Delta_N) \\ \downarrow & & \downarrow \\ (S_n, \emptyset) & \xrightarrow{\pi} & (N, \emptyset) \end{array}$$

Vertical maps are set theoretically the identity;

however, in the orbifold category they are non-trivial Galois coverings with trivial Galois group. There is a non-trivial Galois covering on the level of the local uniformizing neighborhoods with a non-trivial local Galois group.

On $(S_n, \Delta_{\mathbf{m},N})$ one can construct explicit orbifold Kähler metrics (g, ω) that each are determined by smooth functions $\Theta : (-1, 1) \to \mathbb{R}^+$ satisfying certain boundary conditions.
If g_{base} has constant scalar curvature we call such metrics/classes admissible.
The Kähler classes depend on the choice of r from above.

The point of this is:

Due to work by the many of the people mentioned above, one can now write up the appropriate ODE for say admissible KE and CSC where KE comes with a further assumption that g_{base} is KE.

Remark:

The Calabi toric metrics by E. Legendre are special cases - see also recent work by Apostolov, Calderbank, Gauduchon, and Legendre.

The $S^3_{\mathbf{w}}$ join construction

History/Comment:

- Developed generally by (Boyer, Galicki, Ornea)
- is the analogue of Kähler products.

The ingredients in the construction are as follows:

- A compact quasi-regular Sasaki manifold, M, over (N, Δ_N) such that $c_1^{orb}(M/N) = [\omega_N] \in H^2(N, \mathbb{Q})$ is a primitive orbifold Kähler class.
- $c_1^{orb}(S^3_{\mathbf{w}}/\mathbb{CP}^1_{\mathbf{w}}) = [\omega_{\mathbf{w}}] = [\frac{\omega_{FS}}{w^0 w^\infty}]$, where ω_{FS} is the Kähler form of the standard Fubini-Study metric on $\mathbb{CP}^1$ such that $[\omega_{FS}] \in H^2(\mathbb{CP}^1, \mathbb{Z})$ is primitive and $\omega_{\mathbf{w}}$ is the transverse extremal Kähler form on $\mathbb{CP}^1_{\mathbf{w}}$ of the canonical extremal Sasaki structure on the weighted sphere $S^3_{\mathbf{w}}$.
- $w^\infty < w^0$ (for convenience)
- $L = \frac{1}{2l^0}\xi_M - \frac{1}{2l^\infty}\xi_{\mathbf{w}}$ for a pair of relatively prime positive integers l^0, l^∞

- We form the (l^0, l^∞)-join of M and $S^3_{\mathbf{w}}$, $M_{\mathbf{l},\mathbf{w}} = M \star_{\mathbf{l}} S^3_{\mathbf{w}}$, by taking the quotient by the action induced by L:

$$\begin{array}{ccc} M \times S^3_{\mathbf{w}} & & \\ & \searrow \pi_L & \\ \Big\downarrow \pi_2 & & M_{\mathbf{l},\mathbf{w}} \\ & \swarrow \pi_1 & \\ (N, \Delta_N) \times \mathbb{CP}^1_{\mathbf{w}} & & \end{array}$$

- π_2 is the product of the projections of the standard Sasakian projections $\pi_M : M \longrightarrow (N, \Delta_N)$ and $S^3_{\mathbf{w}} \longrightarrow \mathbb{CP}^1_{\mathbf{w}}$.
- $M_{\mathbf{l},\mathbf{w}}$ is a S^1-orbibundle (generalized Boothby-Wang fibration).
- $M_{\mathbf{l},\mathbf{w}}$ has a natural quasi-regular Sasakian structure for all relatively prime positive integers l^0, l^∞. Fixing l^0, l^∞ fixes the contact orbifold.

Lemma (Using Proposition 7.6.6 in Boyer-Galicki book)

If $(M, \mathcal{S})$ *is a quasi-regular Sasaki manifold of order* $\Upsilon_{\mathcal{S}}$*, then the join* $M_{\mathbf{l},\mathbf{w}} = M \star_{\mathbf{l}} S^3_{\mathbf{w}}$ *is smooth if and only if* $\gcd(l^\infty \Upsilon_{\mathcal{S}}, l^0 w^0 w^\infty) = 1$ *where the order* $\Upsilon_{\mathcal{S}}$ *is precisely the order* Υ_N *of the quotient orbifold* (N, Δ_N)*, i.e.* $\Upsilon_{\mathcal{S}} = \Upsilon_N$.

The w-Sasaki cone

- The unreduced Sasaki cone $\mathfrak{t}^+_{M,\mathbf{l},\mathbf{w}}$ of the join $M_{\mathbf{l},\mathbf{w}}$ has a 2-dimensional subcone $\mathfrak{t}^+_{\mathbf{w}}$ is called the **w**-Sasaki cone.
- $\mathfrak{t}^+_{\mathbf{w}}$ is inherited from the Sasaki cone on S^3
- We are interested in quasi-regular Sasakian structures $\mathcal{S}$ in $\mathfrak{t}^+_{\mathbf{w}}$, that is, Reeb vector fields $\xi_{\mathbf{v}}$ that lie on the integer lattice $\Lambda_{\mathbf{w}} \subset \mathfrak{t}^+_{\mathbf{w}}$. They are completely determined by the pair (v^0, v^∞) of relatively prime positive integers.
- To get the orbifold quotients we needed to generalize a result from 2015/16 to the case where $(M, \mathcal{S})$ is not necessarily regular, but still quasi-regular.
- It took some non-trivial nitty-gritty work, but at the end of the day we arrived at...

Theorem (Key Proposition (Boyer, T-F))

Let $(M,\mathcal{S})$ be a quasi-regular Sasakian structure and $M_{\mathbf{l},\mathbf{w}} = M \star_{\mathbf{l}} S^3_{\mathbf{w}}$ the $S^3_{\mathbf{w}}$ join of M. Then the quotient of $M_{\mathbf{l},\mathbf{w}}$ by the S^1 action generated by a quasi-regular Reeb vector field $\xi_{\mathbf{v}} \in \mathfrak{t}^+_{\mathbf{w}}$ is $(S_n, \Delta_{\mathbf{m},N}) \stackrel{\pi^{orb}}{\longrightarrow} (N, \Delta_N)$ from before, where

- $m = \frac{l^\infty}{s}, \quad n = l^0 \frac{(w^0 v^\infty - w^\infty v^0)}{s}, \quad s = \gcd(l^\infty, |w^0 v^\infty - w^\infty v^0|)$.
- $\gcd(m,n) = 1$.
- *The induced primitive (by rescale) orbifold Kähler form $\omega_{n,\mathbf{m},N}$ on the orbifold $(S_n, \Delta_{N,\mathbf{m}})$ satisfies $[\omega_{n,\mathbf{m},N}] = \frac{s}{4\pi \gcd(s\Upsilon_N, w^0 v^\infty l^0) m v^0 v^\infty}[\omega]$, where $[\omega]$ is the admissible Kähler class from above with $r = \frac{w^0 v^\infty - w^\infty v^0}{w^0 v^\infty + w^\infty v^0}$.*

We also get:

- Let $\mathcal{S}_{\mathbf{v}} = (\xi_{\mathbf{v}}, \eta_{\mathbf{v}}, \Phi, g_{\mathbf{v}})$ be a quasiregular Sasakian structure in the **w** cone $\mathfrak{t}^+_{\mathbf{w}}$. The order Υ of $\mathcal{S}_{\mathbf{v}}$ is the product $m v^0 v^\infty \Upsilon_N$.

Sasaki-Einstein

Suppose (N, Δ_N) is positive Kähler-Einstein with Fano index $\mathcal{I}_N$, i.e. $c_1^{orb}(N, \Delta_N) = \mathcal{I}_N[\omega_N]$ and $l^0 = \frac{\mathcal{I}_N}{\gcd(w^0+w^\infty, \mathcal{I}_N)}$, $\quad l^\infty = \frac{w^0+w^\infty}{\gcd(w^0+w^\infty, \mathcal{I}_N)}$, (ensures that $c_1(\mathcal{D})$ vanishes).

- Then $c_1^{orb}(S_n, \Delta_{\mathbf{m},N}) = \left(\frac{v^0+v^\infty}{s}\right)\gcd(l^0 w^0 v^\infty, s\Upsilon_N)[\omega_{n,\mathbf{m},N}]$ so
- the Fano index of $[\omega_{n,\mathbf{m},N}]$ is in turn $\mathcal{I}_{(S_n,\Delta_{\mathbf{m},N})} = \left(\frac{v^0+v^\infty}{s}\right)\gcd(l^0 w^0 v^\infty, s\Upsilon_N)$.
- and coming from the admissible KE ODE we have that $\mathcal{S}_{\mathbf{v}}$ is η-Einstein (up to isotopy) and so has a quasi-regular Sasaki-Einstein structure somewhere in its ray, iff

$$\int_{-1}^{1} \left((1-b)-(1+b)\mathfrak{z}\right)\left((b+t)+(b-t)\mathfrak{z}\right)^{d_N} d\mathfrak{z} = 0, \qquad (1)$$

where $t = w^\infty/w^0$ $(0 < t < 1)$ and $b = v^\infty/v^0$.

The Iterations:

- Guantlett, Martelli, Sparks, Waldram found all the smooth solutions, $Y^{p,q} = S^3 \star_{l,p} S^3_{\frac{p+q}{l},\frac{p-q}{l}}$, where $l = \gcd(p+q, p-q)$ and $N = \mathbb{CP}^1$.
- Building on this: start with a quasi-regular $Y^{p,q}$ and then iterate (in a non-trivial way) the join process again and again - solving (1) at each stage with evolving data $l^0, l^\infty, w^0, w^\infty$ as above - to produce some different kinds of examples.
- At each step, to get **smooth** Sasaki-Einstein examples we need to satisfy (using Lemma 1 above) $\gcd(l^\infty \Upsilon, l^0 w^0 w^\infty) = 1$ where the order Υ increases as we go along.
- We view the round Sasaki Einstein metric on S^3 as a Stage 1 join (since the Kähler quotient is a Stage 1 so-called Bott manifold) and the $Y^{p,q}$ as Stage 2 (Kähler quotient is a Stage 2 Bott orbifold). Here is an example taking one more step to Stage 3:

Example

For all $t \in \mathbb{Z}^+$, $\mathcal{S}_3^t = Y^{780300t^2+65790t+1387,15(170t+7)(306t+13)} \star_{306\,t+13,4} S^3_{17,3}$ has a smooth quasi-regular Sasaki-Einstein structure. The quasi-regular quotient is a "Stage 3 Bott orbifold" given by the log pair $(M_3(a,b,c),\Delta_{\mathbf{m}})$ where $\Delta_{\mathbf{m}} = \sum_{i=1}^3 \left(\left(1-\frac{1}{m_i^0}\right)D_{u_i^0} + \left(1-\frac{1}{m_i^\infty}\right)D_{u_i^\infty}\right)$

$$
\begin{aligned}
a &= 6(255\,t+11)(510\,t+21)(1020\,t+43) \\
b &= 204(255\,t+11)(765\,t+32)(1020\,t+43)(306\,t+13) \\
c &= 51(306\,t+13) \\
\mathbf{m} &= (m_1^0, m_1^\infty, m_2^0, m_2^\infty, m_3^0, m_3^\infty) \\
(m_1^0, m_1^\infty) &= (1,1) \\
(m_2^0, m_2^\infty) &= (1387+65790t+780300t^2)(1020\,t+43, 2(255\,t+11)) \\
(m_3^0, m_3^\infty) &= 2(17,9).
\end{aligned}
$$

A quick note on Bott manifolds

- First introduced by Grossberg and Karshon.
- In a joint paper with Boyer and Calderbank we studied the smooth cases.
- Stage 1, M_1 is just $\mathbb{CP}^1$
- Stage 2, $M_2(a)$ is a Hirzebruch surface $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(a)) \to M_1$
- Stage 3, $M_3(a, b, c)$ is the total space of a bundle $\mathbb{P}(\mathbb{1} \oplus \mathcal{L}) \to M_2(a)$ where b, c determines the Chern class of $\mathcal{L}$ ...and so on...
  Then $D_{u_i^0}$ and $D_{u_i^\infty}$ mentioned above are the natural exceptional divisors coming from the zero and infinity sections at each stage.

Natural Question:

Do all Stage 3 Bott orbifolds arise as quasi-regular quotients via iterated joins?
Answer: Nope! E.g. the Koiso-Sakane manifold $M(0,1,-1) = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(1,-1)) \to \mathbb{CP}^1$ cannot arise this way. (Since $c_1(\mathcal{O}(1,-1))$ is not $\pm$ a Kähler class.)

By the way:

$M(0,1,-1)$ does arise as a regular quotient of a so-called Yamazaki fiber join. The corresponding Sasaki structure here is CSC (up to isotopy) but not Sasaki Einstein...This is a different story (also joint with Boyer) for another time.

Perpetuity?

We can iterate on the example above a few times (tiresome and messy!!!) to get:

Proposition:

In dimensions 9 and 11 there exist countably infinite families of quasi-regular smooth Sasaki Einstein structures in the form of non-trivial iterated $S^3_{\mathbf{w}}$-joins $M^{2k+1}_{\mathbf{l},\mathbf{w}}$

Conjecture:

There are iterated $S^3_{\mathbf{w}}$ joins (with no trivial iterations) admitting smooth Sasaki-Einstein structures in all odd dimensions.

Remark:

Due to the smoothness condition this is not just an easy induction...

To find out more:

- **Boyer and Galicki** Sasakian geometry, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.
- **Apostolov, Calderbank, Gauduchon, and T-F.** Hamiltonian 2-forms in Kähler geometry, III *Extremal metrics and stability*, Inventiones Mathematicae 173 (2008), 547–601.
- **Boyer and T.-F.** The Sasaki Join, Hamiltonian 2-forms, and Constant Scalar Curvature , J Geom Anal (2016) 26, 1023–1060
- **Boyer, Calderbank, and T.-F.** The Kähler geometry of Bott manifolds, Advances in Mathematics, 350 (2019), 1–62
- **Boyer and T.-F.** Sasaki-Einstein Metrics on a class of 7-Manifolds, Journal of Geometry and Physics, Volume 140, June 2019, 111–124
- recent preprints of **Boyer and T.-F.**
  - The $S^3_{\mathbf{w}}$ Sasaki Join Construction, arXiv:1911.11031
  - Iterated S^3 Sasaki Joins and Bott Orbifolds, arXiv:2006.06596
  - Sasakian Geometry on Sphere Bundles, arXiv:2007.10236 (this was for the "By the way")

THANK YOU!