Variational problems in conformal geometry

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03.07.2020
CIRGET Geometry & Topology Seminar

Almost Hermitian metrics

- Let (X^{2n}, J) be an almost complex manifold, $n > 1$, and g an almost Hermitian metric, i.e. $g(J\cdot, J\cdot) = g(\cdot, \cdot) \rightsquigarrow \Omega := g(J\cdot, \cdot)$ positive $(1,1)$-form

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$$d\Omega^{n-1} = \theta \wedge \Omega^{n-1}$$

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Theorem (Gauduchon '77)

Let (X^{2n}, J) be a compact almost complex manifold of real dimension $2n > 2$ and let c be a conformal class of almost Hermitian metrics on X. Then there exists a Gauduchon metric $g \in c$. Moreover, g is unique up to multiplication by a constant.

Questions

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- Find *interesting* metrics as extremals of natural functionals.

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Gauduchon '84: $\mathcal{L}_G : \mathcal{H}_1 \to \mathbb{R}, \ \mathcal{L}_G(\Omega) = \int_X |\theta|^2 dv_\Omega$

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Theorem (Angella, I, Otiman, Tardini '20)

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Proof

- Ω is a critical point iff $\frac{d}{dt}|_{t=0}\mathcal{G}(\Omega_t) = 0$, for any variation $\Omega_t = \varphi_t\Omega \in \{\Omega\}_1$ with $\Omega_0 = \Omega$.

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$\rightsquigarrow$ $d^*\theta(x_0) = 0$. As $\int_X d^*\theta dv_\Omega = 0 \Rightarrow d^*\theta \equiv 0$ so Ω is Gauduchon. ■

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The form $dJ\theta$

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  and one has $\rho^B = \rho^{Ch} + dJ\theta$, where $\rho =$ the Ricci form of the connection$\equiv$ the curvature induced on $\det TX$.

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- If $d\theta = 0$ (i.e. $\{\Omega\}$ is lcK or lcb), then

$$[\theta]_{dR} \in H^1(X, \mathbb{R}) \cong \mathrm{Hom}(\pi_1(X), \mathbb{R}_{>0}) \ni \rho$$

defines a flat holomorphic line bundle

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and $\{\Omega\} \in \mathcal{C}^\infty(X, S^2T^*X \otimes (\mathcal{L}^*)^{1/n-1})$.

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The form $dJ\theta$

- If $d\theta = 0$ (i.e. $\{\Omega\}$ is lcK or lcb), then

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The functional $\mathcal{F}$

Let (X^{2n}, J) be a compact almost complex manifold. Fix $\{\Omega\}_1$ a conformal class of normalised almost Hermitian metrics on (X, J) and consider:

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Proposition (Angella, I, Otiman, Tardini '20)

Let (X, J) be a compact almost complex $2n$-dimensional manifold. Then Ω is a critical point of $\mathcal{F}$ if and only if

$$(n-2)|dJ\theta_\Omega|^2 - 2(n-1)(dd^c)^*\Delta\Omega = k, \quad k \in \mathbb{R}.$$

Minima of $\mathcal{F}$ on complex surfaces

Theorem (Angella, I, Otiman, Tardini '20)

Given a compact complex surface (X, J) and a conformal class of normalized Hermitian metrics $\{\Omega\}_1$, there exists and is unique an extremal metric $\Omega \in \{\Omega\}_1$ for the functional $\mathcal{F}$ restricted to $\{\Omega\}_1$. Moreover, $\mathcal{F}(\Omega)$ is an absolute minimum for $\mathcal{F}$ on $\{\Omega\}_1$.

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- each Ω_s minimizes $\mathcal{F}$ on $\{\Omega_s\}_1$
- however $0 \notin \operatorname{im} \mathcal{F}$ since X admits no Kähler metric ($b_1(X) = 1$)

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Hence Ω is critical for $\mathcal{F}$ iff $dd^c(\Lambda(dJ\theta)\Omega) = 0$.

Distinguished metrics

Definition

Let (X^{2n}, J) be an almost complex manifold and Ω an almost Hermitian metric. We call Ω *distinguished* if

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Proof

$$
\begin{aligned}
dd^c(q\Omega^{n-1}) &= \Lambda(dd^c q - d^c q \wedge \theta + dq \wedge J\theta + qdJ\theta + q\theta \wedge J\theta)\frac{\Omega^n}{n} \\
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Fix $\Omega_0 \in c$, and let L_0, L_0^* denote the same operators w.r.t. Ω_0.

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$$Lq = \mathrm{e}^{-n\varphi} L_0(q\mathrm{e}^{(n-1)\varphi})$$
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- We are looking for $\varphi \in \mathcal{C}^\infty(X)$ and $k \in \mathbb{R}$ s.t.

$$(f_0 + (n-1)L_0^*\varphi)^{n-1} = k^{n-1}\alpha \Leftrightarrow (n-1)L_0^*\varphi = k\alpha^{1/n-1} - f_0$$

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- Thus $\exists \varphi \in \mathcal{C}^\infty(X)$ with $L f^{n-1}_{\mathrm{e}^\varphi \Omega_0} = 0$ iff $k = \frac{\int_X f_0 \alpha dv_0}{\int_X \alpha^{n/n-1} dv_0}$.

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  - $k > 0 \Rightarrow f_\Omega > 0$ and $f_\Omega\Omega$ is Gauduchon. ■

Distinguished versus Gauduchon metrics

Corollary

Let (X, J, Ω) be a compact almost complex manifold endowed with an almost Hermitian metric. Then any two of the following affirmations imply the third one:

1. Ω is Gauduchon
2. Ω is distinguished
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$\rightsquigarrow$ on manifolds with $\chi(X) \neq 0$, distinguished$\neq$ Gauduchon unless balanced ($\theta = 0$).

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But $\int_X q dv = 0 \rightsquigarrow q \equiv 0$ so Ω is Gauduchon. ■

The functional $\mathcal{A}$

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$$\mathcal{A} : \{\Omega\}_1 \to \mathbb{R}, \quad \mathcal{A}(\Omega) := \int_X |d^c\Omega|^2 dv_\Omega$$

Proposition (Angella, I, Otiman, Tardini '20)

The extremal metrics for the functional $\mathcal{A}$ are characterized by the equation:

$$(n-1)|d\Omega|^2 + 2d^*\theta = k, \quad k \in \mathbb{R}.$$

The functional $\mathcal{R}$

Let (X^{2n}, J) be a compact almost complex manifold and let $\{\Omega\}_1$ be a conformal class of normalised almost Hermitian metrics.

$$\mathcal{R} : \{\Omega\}_1 \to \mathbb{R}, \quad \mathcal{R}(\Omega) := \int_X |dd^c\Omega|^2 dv_\Omega$$

Proposition (Angella, I, Otiman, Tardini '20)

The extremal metrics for the functional $\mathcal{R}$ are characterized by the equation:

$$(n-4)|dd^c\Omega|^2 + 2\Lambda(dd^c)^* dd^c\Omega = k, \quad k \in \mathbb{R}.$$