CORRECTION TO THE PAPER *BIHERMITIAN STRUCTURES ON COMPLEX SURFACES*, PROC. LONDON MATH. SOC. 79 (1999), 414–428

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Abstract. We here correct two statements, Theorem 1 and Corollary 2, of the above-mentioned paper and provide a new argument for Corollary 2. We also update some parts of the paper in the light of recent major new developments in the theory of bihermitian structures due to N. Hitchin.

1. Theorem 1

Part (ii) of Theorem 1 contains a mistake which has been spotted by N. Hitchin. The original statement of Theorem 1 must then be replaced by the following one.

Theorem 1. *Let (M, c, J_1, J_2) be a connected, compact bihermitian conformal 4-manifold with even first Betti number. Then,*

(i) *either (c, J_1, J_2) is strongly bihermitian, in which case the complex surfaces (M, J_1) and (M, J_2) are either both complex tori or both* K3 *surfaces, or*
(ii) *(M, J_1) (or equivalently (M, J_2)) is either $\mathbb{C}P^2$ or a minimal ruled surface admitting an effective anti-canonical divisor, or a surface obtained from them by blowing up points along an effective anti-canonical divisor.*

Our mistake was the erroneous assertion appearing on page 420 of our paper that *a minimal ruled surface has a Kähler metric of negative total scalar curvature* (*and hence admits no holomorphic sections of the anti-canonical bundle*) *as soon as its genus is greater than 1.* This is evidently false as shown by the following simple counter-example communicated by N. Hitchin: consider a Riemann surface C of any genus $\mathbf{g}$ and let $K = T^*C$ be the canonical bundle over C; then, the inverse of the holomorphic Liouville 2-form on the total space of K extends to a non-trivial holomorphic section of the anti-canonical bundle of the corresponding minimal ruled surface $\mathbb{P}(K \oplus \mathcal{O})$.

Notice that ruled surfaces admitting a non-trivial holomorphic section of the anti-canonical line bundle — equivalently a non-trivial holomorphic Poisson structure — are well understood thanks to F. Sakai's paper [**Sa**] and the recent work by C. Bartocci and E. Macrì [**BM**].

Remark 1. A major new development of the theory of bihermitian structures has been its re-interpretation within the general framework of generalized Kähler structures recently developed by M. Gualtieri [**Gu**] and N. Hitchin [**Hi1**], and the explicit construction by N. Hitchin of $SU(2)$-invariant bihermitian structures on the projective space $\mathbb{C}P^2$ and on the product $\mathbb{C}P^1 \times \mathbb{C}P^1$ [**Hi2**]. This gives a positive answer to a question which had been left open in our paper and was pointed out as a main issue in the theory, namely the existence of examples in Theorem 1-(ii).

2. Corollary 2

Corollary 2 omits one family of complex surfaces, namely the *hyperbolic Inoue surfaces* according to Nakamura's terminology in [**Na**]. Our mistake stems from a careless reading of Reference [**19**] (loc. cit.) and the false inference that the anti-canonical divisors of J_1 and J_2 must be *homologically* the same. We here provide a corrected statement of Corollary 2 as well as an alternative argument using Nakamura's paper [**Na**].

Corollary 2. *Let (M, c, J_1, J_2) be a connected, compact bihermitian surface with odd first Betti number and assume that the conformal class c contains a metric g such that $\theta_1 + \theta_2 = 0$. Then (c, J_1, J_2) is* not *strongly bihermitian and (M, J_i) is either a Hopf surface, or a parabolic Inoue surface, or a hyperbolic Inoue surface, or a complex surface obtained from them by blowing up points situated on some anti-canonical divisor. In particular, up to a finite cover, M is diffeomorphic to $(S^1 \times S^3)\sharp k\overline{\mathbb{C}P}^2$ for some $k \geq 0$.*

Proof. By Lemma 3 both complex surfaces (M, J_1) and (M, J_2) have an effective anti-canonical divisor supported on the union of $\mathcal{D}^+$ and $\mathcal{D}^-$. By their very definition, these are disjoint — recall that $\mathcal{D}^+$, resp. $\mathcal{D}^-$, has been defined as the set of points where $J_1 = J_2$, resp. $J_1 = -J_2$ — and, by Proposition 4, they are both non-empty (beware however that the anti-canonical divisors of (M, J_1) and (M, J_2) are homologically distinct in general, since J_1 and J_2 induce opposite orientations on $\mathcal{D}^-$). It follows that the Kodaira dimensions of (M, J_1) and (M, J_2) are $-\infty$, i.e. they belong to the Class VII in the Enriques–Kodaira classification. The blow-up formula for the anti-canonical bundle then shows that the minimal model of (M, J_1) is again a surface of the Class VII admitting an effective anti-canonical divisor with at least two connected components, and that all the blown up points — if any — lie on this divisor, and similarly for the minimal model of (M, J_2) (compare with Cor. 3.14 in Reference [**19**] of the paper).

We now show that a minimal surface, say S, of the Class VII which admits an effective anti-canonical divisor, D, with at least two connected components is either a Hopf surface, or a parabolic Inoue surface, or a hyperbolic Inoue surface. This easily follows from Nakamura's paper [**Na**] and we only outline argument, referring to [**Na**] for details and notation. By a well-known result of K. Kodaira — see e.g. [**BHPV**, Thm. 18.6] — a minimal surface of the Class VII is either a Hopf surface or of algebraic dimension 0: we may then assume that the algebraic dimension of S is 0, which is the overall assumption in [**Na**]. From Lemma (2.2) in [**Na**] and from the first exact sequence in the proof of this lemma, we get $h^1(D, \mathcal{O}_D) = 2$ as D has at least two connected components. From Lemmas (2.8) and (2.12) in [**Na**] we then infer that $D = D_1 + D_2$, where D_1 and D_2 are connected and pertain to one of the following three cases:

- either D_1 and D_2 are both non-singular elliptic curves: then S must be a Hopf surface by [**Na**, Thm. 5.2], or
- D_1 is a non-singular elliptic curve and D_2 is a cycle of rational curves: then S is a parabolic Inoue surface by [**Na**, Thm. 7.1], or
- D_1 and D_2 are both cycles of rational curves: then S is a hyperbolic Inoue surface by [**Na**, Thm. 8.1].

The last assertion of Corollary 2 follows from Corollary (12.6) in [**Na**]. □

Remark 2. Since the appearance of our paper, more results have been obtained concerning bihermitian structures on compact complex surfaces with odd first Betti number, especially in [**Ap**] and in [**Dl**].

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