

# BIHERMITIAN STRUCTURES ON COMPLEX SURFACES

V. APOSTOLOV, P. GAUDUCHON *and* G. GRANTCHAROV

[Received 13 March 1998—Revised 14 September 1998]

## 1. Introduction

We consider connected, oriented (Riemannian) conformal 4-manifolds  $(M, c)$  admitting two independent compatible integrable almost-complex structures  $J_i$ , for  $i = 1, 2$ . Here and henceforth, *compatible* means that  $J_1$  and  $J_2$  are orthogonal with respect to any Riemannian metric  $g$  in the conformal class  $c$  and induce the chosen orientation of  $M$ , and *independent* means that  $J_1(x) \neq \pm J_2(x)$  for some point  $x$  in  $M$ . Then,  $(c, J_1, J_2)$  will be called a (conformal) *bihhermitian* structure on  $M$  and  $(M, c, J_1, J_2)$  a *bihhermitian surface*.

A bihermitian structure will be called *strongly bihermitian* if  $J_1(x) \neq \pm J_2(x)$  everywhere.

In the sequel, an integrable almost-complex structure will occasionally be called a *complex structure*.

A Riemannian conformal structure  $c$  is called *anti-self-dual*, ASD for short, if the self-dual part  $W^+$  of the Weyl tensor vanishes identically. It is an easy consequence of Theorem 4.1 in [5] that  $c$  is ASD if and only if the following holds: *for any point  $x$  of  $M$ , any compatible complex structure of the tangent space  $T_x M$  can be extended to a compatible complex structure on some neighbourhood of  $x$* . Thus, ASD conformal 4-manifolds locally admit infinitely many compatible complex structures, but it may happen that no one of them extends to the whole manifold, as in the case of the standard conformally flat 4-sphere  $S^4$ . Kähler complex surfaces with zero scalar curvature provide examples of ASD Hermitian 4-manifolds; conversely, any ASD Hermitian metric on a *compact* 4-manifold is locally conformal to a Kähler metric with zero scalar curvature [7].

Compact ASD bihermitian surfaces have recently been classified by M. Pontecorvo [19]. Among them, strongly bihermitian surfaces exactly correspond to *hyperhermitian complex surfaces*  $(M, c, \mathcal{F})$ , where  $\mathcal{F}$  is a 2-sphere of compatible complex structures generated by two *anti-commuting* ones; these are (see [8]) flat complex tori and Ricci flat K3 surfaces, which are *hyperkähler*, and some class of conformally flat Hopf surfaces.

Certain conformally flat Hopf surfaces actually admit *two* distinct hyperhermitian structures, say  $(c, \mathcal{F})$  and  $(c, \mathcal{F}')$ , compatible with the same (conformally flat) conformal structure and the same orientation; see for example, [12] or [19]. We thus get two sorts of bihermitian structures  $(c, J_1, J_2)$  with respect to the same (flat) conformal metric  $c$ , according to whether or not  $J_1$  and  $J_2$  belong to the same sphere  $\mathcal{F}$  or  $\mathcal{F}'$ . In the former case,  $(c, J_1, J_2)$  is strongly

bihhermitian, whereas it is *not* in the latter case. More precisely, the subsets where  $J_1$  is equal or conjugate to  $J_2$  are then both non-empty; compare with Proposition 2 below. Other ASD bihermitian surfaces appearing in Pontecorvo's list are obtained from either a Hopf surface or a parabolic Inoue surface by blowing up points along a unique elliptic curve [19, Theorem 5.2] (see also Corollary 3 below), that is, they belong to the class VII in the Kodaira classification; in particular, their first Betti number is equal to 1.

In this paper, we are mostly concerned with the *non-ASD* case. Non-ASD connected conformal 4-manifolds admit at most *two* independent compatible complex structures [19, Corollary 1.6] so that bihermitian structures are also 'maximally' hermitian. Moreover, non-ASD bihermitian situations cannot occur in the case when  $W^+$  is everywhere *degenerate*, that is, has only two distinct eigenvalues as an operator acting on the bundle  $\Lambda^+M$  of self-dual 2-forms. (For that and for further information concerning the interplay between conformal and complex geometry in the non-ASD case, see for example, [9, 20, 3].) In particular, if  $(c, J_1, J_2)$  is a non-ASD bihermitian structure, neither  $J_1$  nor  $J_2$  can be Kähler with respect to any metric in the conformal class  $c$ .

It has been expected for a long time that bihermitian situations could not possibly occur, even locally, in the non-ASD case. It thus came as a (welcome) surprise when P. Kobak [16] recently gave a general and explicit method to produce non-ASD (strongly) bihermitian structures (called *doubly-Hermitian* in [16]) on  $\mathbb{R}^4$  or the torus  $T^4$ .

The present paper includes elements of a general theory of (non-ASD) bihermitian 4-manifolds opened up by Kobak's paper and a series of results aiming at a classification in the compact case (still to come), principally in the case when the first Betti number is even. We also provide a general method, partially suggested by D. Joyce, for constructing non-ASD strongly bihermitian conformal structures on (compact) surfaces admitting a hyperkähler metric, that is, on tori and K3 surfaces. A similar procedure also provides non-ASD strongly bihermitian structures on  $S^1 \times S^3$ .

The paper is organized as follows.

Section 2 is principally devoted to proving the following theorem.

**THEOREM 1.** *Let  $(M, c, J_1, J_2)$  be a connected, compact bihermitian conformal 4-manifold with even first Betti number. Then,*

- (i) *either  $(c, J_1, J_2)$  is strongly bihermitian, in which case the complex surfaces  $(M, J_1)$  and  $(M, J_2)$  are either both complex tori or both K3 surfaces, or*
- (ii)  *$(M, J_1)$  (or equivalently  $(M, J_2)$ ) is obtained from either  $\mathbb{C}P^2$  or a minimal ruled surface of genus at most 1 by blowing up points along some anti-canonical divisor.*

No examples of bihermitian structures on complex surfaces described in Theorem 1(ii) are known at present. If such examples were found, they would be the first known examples of bihermitian structures on a complex surface *not* admitting any ASD metric and would thus constitute a major step in the development of the theory.

An immediate consequence of Theorem 1 is the following generalization (in complex dimension 2) of the well-known results of Lichnerowicz [17] for elements in the connected group of isometries of a compact Kähler manifold.

**COROLLARY 1.** *Let  $(M, g, J)$  be a compact Hermitian surface with even first Betti number. Suppose that there exists an orientation-preserving conformal isometry of  $(M, g)$  which is not  $\pm$ -biholomorphic. Then  $(M, J)$  is one of the complex surfaces described in Theorem 1.*

A ‘Kählerian’ version of the above corollary says that the positive conformal isometries of any compact Kähler, non-hyperkähler surface preserve the Kähler structure [19, Theorem 5.3] (see also [4, Theorem 2]). On the other hand, it is not difficult to find orientation-preserving isometries on flat tori which are not  $\pm$ -holomorphic. For some special K3 surfaces such isometries have been shown to exist; see [1]. Outside the Kählerian category, we can construct (non-Kähler) Hermitian conformal metrics on some complex tori that admit an orientation-preserving conformal isometry which does not preserve the complex structure (see §4.1 below). Notice that locally conformally Kähler surfaces with odd first Betti number, admitting orientation-preserving isometries which are not  $\pm$ -biholomorphisms, are anti-self-dual bihermitian surfaces, so they are either the conformally flat Hopf surfaces or obtained from a Hopf surface or a parabolic Inoue surface by blowing up points along a unique smooth elliptic curve; cf. [19, Theorem 5.2].

In §3, we investigate the situation described in Theorem 1(i) and show that (non-ASD) bihermitian structures on complex tori or K3 surfaces, which are necessarily strongly bihermitian by Theorem 1, actually determine, and are determined by, a special class of *twisted* hyperkähler structures; more precisely, we prove the following.

**THEOREM 2.** *Let  $M$  be either a 4-dimensional torus or a K3 surface. Then, for any bihermitian structure  $(c, J_1, J_2)$  on  $M$ , there exist a triple of real symplectic 2-forms,  $\Phi_0, \Phi_1, \Phi_2$ , such that the complex 2-forms  $\Omega_1 = \Phi_0 - i\Phi_1$  and  $\Omega_2 = \Phi_0 - i\Phi_2$  are holomorphic symplectic 2-forms with respect to  $J_1$  and  $J_2$  respectively. Moreover,  $\Phi_0, \Phi_1$  and  $\Phi_2$  are related by*

- (a)  $\Phi_0 \wedge \Phi_0 = \Phi_1 \wedge \Phi_1 = \Phi_2 \wedge \Phi_2$ ,
- (b)  $\Phi_0 \wedge \Phi_1 = \Phi_0 \wedge \Phi_2 = 0$ ,
- (c)  $\Phi_1 \wedge \Phi_2 = -p \Phi_0 \wedge \Phi_0$ ,

where the smooth function  $p = -\frac{1}{4} \text{trace}(J_1 \circ J_2)$  satisfies  $|p| < 1$ .

Conversely, let  $M$  be a 4-dimensional manifold equipped with a triple of real symplectic forms,  $\Phi_0, \Phi_1, \Phi_2$  that satisfy the conditions (a), (b), (c) for some smooth function  $p$  with  $|p| < 1$ . Let  $M$  be oriented by any one of these symplectic forms. Then, the triple  $\Phi_0, \Phi_1, \Phi_2$  determines a conformal strongly bihermitian structure  $(c, J_1, J_2)$  such that  $\Phi_0, \Phi_1$  and  $\Phi_2$  are self-dual with respect to  $c$  and the orientation, and the complex 2-forms  $\Omega_1 = \Phi_0 - i\Phi_1$  and  $\Omega_2 = \Phi_0 - i\Phi_2$  are holomorphic with respect to  $J_1$  and  $J_2$  respectively. Moreover,  $(c, J_1, J_2)$  is hyperkähler if and only if  $p$  is constant.

In §4, we show how Theorem 2 can be used as an effective tool for constructing bihermitian structures. We first recover Kobak’s constructions, then, following an idea of D. Joyce, we propose a general procedure for constructing non-anti-self-dual (strongly) bihermitian structures by deforming hyperkähler structures on complex tori or K3 surfaces. A similar construction involving the

flat hyperkähler structure of  $\mathbb{C}^2$  provides a non-ASD bihermitian conformal class on any Hopf surface admitting a hyperhermitian metric. Bringing together these constructions with the above-mentioned classification of compact hyperhermitian complex surfaces by C. Boyer, we eventually get the following.

**PROPOSITION 1.** *Any hyperhermitian structure on a compact complex surface can be deformed into a non-ASD strongly bihermitian structure.*

The above proposition shows in particular the existence of non-ASD bihermitian structures on  $S^1 \times S^3$ . Apart from that, only a little information is available concerning bihermitian situations for a compact 4-dimensional manifold with *odd* first Betti number. Some results pertaining to this case are collected in § 5.

The *angle function* of a bihermitian structure  $(c, J_1, J_2)$  is the real function  $p$  defined by

$$p = -\frac{1}{4} \text{trace}(J_1 \circ J_2), \quad (1)$$

or, equivalently,

$$J_1 \circ J_2 + J_2 \circ J_1 = -2p \text{Id}, \quad (2)$$

where Id denotes the identity. Notice that the angle-function  $p$  only depends upon the conformal class  $c$ . We then have  $|p| \leq 1$ ,  $p(x) = \pm 1$  if and only if  $J_1(x) = \pm J_2(x)$ , and  $p(x) = 0$  if and only if  $J_1(x)$  and  $J_2(x)$  anticommute. If  $p$  is a constant, of absolute value less than 1,  $J_1$  and

$$J_2' := \frac{1}{\sqrt{1-p^2}} (J_2 - pJ_1)$$

anticommute, and hence determine a 2-sphere  $\mathcal{F}$  of almost-complex structures, all compatible with  $c$  and with the same orientation. Moreover, since  $p$  is constant and  $J_1$  and  $J_2$  are both integrable, *all* almost-complex structures in  $\mathcal{F}$  are integrable as well (see [13, 2]), so that  $(c, \mathcal{F})$  is actually a (conformal) hyperhermitian structure. Then, for any chosen metric  $g$  in  $c$ , all Hermitian structures  $(g, J)$  have the same Lee form  $\theta$  when  $J$  belongs to  $\mathcal{F}$ , and the hyperhermitian structure  $(c, \mathcal{F})$  is (conformal) hyperkähler if  $\theta$  is zero for some metric  $g$  in  $c$ .

In § 6, we investigate further properties of the angle-function  $p$ . We infer in particular that for any strongly conformal bihermitian structure of a compact complex surface, the self-dual Weyl tensor  $W^+$  must vanish somewhere (see [2] for specific properties of  $p$  in the ASD case).

*Acknowledgements.* The first-named author would like to thank the Centre de Mathématiques de l'École Polytechnique, U.R.A. 169 du C.N.R.S. for its generous hospitality. The authors thank P. Kobak, M. Pontecorvo and S. Salamon for their interest and useful remarks. The authors are specially grateful to D. Joyce for substantially improving the material presented in § 4. They also thank the referee for correcting a mistake in the proof of Proposition 3.

## 2. Proof of Theorem 1

For any almost-hermitian structure  $(g, J)$  on a 4-dimensional manifold  $M$ , the *Kähler form* and the *Lee form* of  $(g, J)$  are the real 2-form,  $F$ , and the real 1-form  $\theta$ , defined by  $F = g(J \cdot, \cdot)$  and  $dF = \theta \wedge F$ .

Let  $(c, J_1, J_2)$  be a bihermitian structure on  $M$ .

Fix a metric  $g$  in  $c$  and denote by  $F_1$  and  $F_2$  the Kähler forms of  $(g, J_1)$  and  $(g, J_2)$ . Let  $\theta_1$  and  $\theta_2$  be the corresponding Lee forms.

The following lemma is a direct consequence of [3, Lemma 2 and Remark 1].

LEMMA 1. *For any metric  $g$  in the conformal class  $c$ , the Lee forms of  $J_1$  and  $J_2$  satisfy*

$$(d(\theta_1 + \theta_2))_+ = 0 \quad (3)$$

and

$$2\delta\theta_1 + |\theta_1|^2 = 2\delta\theta_2 + |\theta_2|^2, \quad (4)$$

where, for any 2-form  $\Phi$ ,  $\Phi_+$  denotes the self-dual part of  $\Phi$ ,  $\delta$  denotes the codifferential with respect to  $g$ , and  $|\cdot|$  the norm induced by  $g$ .

Let  $\mathcal{U}$  be the open subset of points  $x$  of  $M$  such that  $J_1(x) \neq \pm J_2(x)$ . Then the angle function  $p$  (see § 1) satisfies  $|p| < 1$  on  $\mathcal{U}$ . Setting  $q = \sqrt{1 - p^2}$ , we consider the positive  $c$ -compatible almost-complex structures  $I_0, I_1, I_2$  on  $\mathcal{U}$  defined by

$$I_1 = \frac{1}{q}(J_2 - pJ_1), \quad I_2 = \frac{1}{q}(J_1 - pJ_2), \quad I_0 = J_1 \circ I_1 = -J_2 \circ I_2. \quad (5)$$

The Kähler forms of  $(g, I_0), (g, I_1)$  and  $(g, I_2)$  are denoted by  $\Phi_0, \Phi_1$  and  $\Phi_2$  respectively.

Since  $J_1$  and  $J_2$  are both integrable, we clearly have  $I_0(\delta\Phi_0) = I_0(J_1\delta\Phi_1) = I_1(\delta\Phi_1)$  and  $I_0(\delta\Phi_0) = -I_0(J_2\delta\Phi_2) = I_2(\delta\Phi_2)$ . It follows that the almost-hermitian structures  $(g, I_0), (g, I_1)$  and  $(g, I_2)$  have the same Lee form, say  $\tau$ .

LEMMA 2. *For any metric  $g$  in the conformal class  $c$  we have (on  $\mathcal{U}$ )*

$$\theta_1 + \theta_2 - 2\tau = -2d \ln q. \quad (6)$$

*Proof.* It follows from the definitions of  $I_0, I_1$  and  $I_2$  that

$$F_1 = pF_2 + q\Phi_2, \quad (7)$$

$$F_2 = pF_1 + q\Phi_1. \quad (8)$$

In the following equalities,  $\nabla$  denotes the Levi-Civita connection of  $g$  and we freely identify vectors and covectors via  $g$ . The integrability of  $J_1$  and  $J_2$  then implies that

$$\nabla F_1 = \frac{1}{2}I_0\theta_1 \otimes \Phi_1 - \frac{1}{2}I_1\theta_1 \otimes \Phi_0, \quad (9)$$

$$\nabla \Phi_1 = -\frac{1}{2}I_0\theta_1 \otimes F_1 + \frac{1}{2}I_1(2\tau - \theta_1) \otimes \Phi_0, \quad (10)$$

$$\nabla F_2 = -\frac{1}{2}I_0\theta_2 \otimes \Phi_2 + \frac{1}{2}I_2\theta_2 \otimes \Phi_0, \quad (11)$$

$$\nabla \Phi_2 = \frac{1}{2}I_0\theta_2 \otimes F_2 + \frac{1}{2}I_2(2\tau - \theta_2) \otimes \Phi_0. \quad (12)$$

By using (7) and (8), we obtain from (9), (10), (11) and (12),

$$df = f(\theta_1 - \tau) + I_0(\theta_1 - \tau),$$

$$df = f(\theta_2 - \tau) - I_0(\theta_2 - \tau),$$

where  $f = p/q$ . By adding and subtracting these relations, we get

$$2df = f(\theta_1 + \theta_2 - 2\tau) + I_0(\theta_1 - \theta_2), \quad (13)$$

$$I_0(\theta_1 + \theta_2 - 2\tau) = f(\theta_2 - \theta_1), \quad (14)$$

whence, finally, we obtain (6).

Let  $[J_1, J_2] = J_1 \circ J_2 - J_2 \circ J_1$  be the commutator of  $J_1$  and  $J_2$ . For any metric  $g$  in the conformal class  $c$ , we consider the real  $J_1$ -anti-invariant 2-form  $\Phi(\cdot, \cdot) = \frac{1}{2}g([J_1, J_2] \cdot, \cdot)$  and the corresponding complex  $(0, 2)$ -form  $\sigma_1(\cdot, \cdot) = \Phi(\cdot, \cdot) + i\Phi(J_1 \cdot, \cdot)$ . Thus,  $\sigma_1$  is a smooth section of  $\Lambda_{J_1}^{0,2}(M)$  and  $\sigma_1(x) = 0$  if and only if  $\Phi(x) = 0$ , which is so if and only if  $J_1(x) = \pm J_2(x)$ . Notice that the complex line bundle  $\Lambda_{J_1}^{0,2}(M)$  is identified, via the metric  $g$ , with the *anti-canonical* bundle  $K_{J_1}^{-1}$ . Moreover, the canonical Hermitian connection of the holomorphic line bundle  $K_{J_1}^{-1} = \Lambda_{J_1}^{0,2}(M)$  coincides with the connection induced by the Chern connection of  $(g, J_1)$ ; both will be denoted by  $\nabla^1$ . We then have the following.

LEMMA 3. *For any  $(0, 1)$ -vector field  $X$  of  $(M, J_1)$ ,*

$$\nabla_X^1 \sigma_1 = -\frac{1}{2}(\theta_1 + \theta_2)(X) \sigma_1. \quad (15)$$

*Proof.* Let  $\mathcal{U}$  be the open subset of  $M$  where  $J_1 \neq \pm J_2$ . According to [19, Proposition 1.3],  $\mathcal{U}$  is dense in  $M$ , so it is sufficient to verify (15) on  $\mathcal{U}$ . It is easily checked that  $\sigma_1 = q(\Phi_0 + i\Phi_1)$  on  $\mathcal{U}$ . So by (9) and (10) we obtain

$$(\nabla_X \sigma_1)^{0,2} = (d \ln q - \tau + \frac{1}{2}\theta_1)(X) \sigma_1, \quad (16)$$

where  $X$  is an arbitrary  $(0, 1)$ -vector field (with respect to  $J_1$ ) and  $(\cdot)^{0,2}$  denotes the  $(0, 2)$  part. Now, by using Lemma 2, we find that the equality (16) reads

$$(\nabla_X \sigma_1)^{0,2} = -\frac{1}{2}\theta_2(X) \sigma_1.$$

On the other hand, it easily follows from the well-known relation between  $\nabla$  and  $\nabla^1$  (cf. for example, [21]) that, for any  $(0, 2)$ -form and  $(0, 1)$ -vector, we have

$$\nabla_X^1 \sigma_1 = (\nabla_X \sigma_1)^{0,2} - \frac{1}{2}\theta_1(X) \sigma_1.$$

Lemma 3 follows directly from the above two equalities.

The above lemma implies that  $\sigma_1$  is a holomorphic section of  $K_{J_1}^{-1}$  provided that  $\theta_1 + \theta_2 = 0$  for some metric  $g$  in  $c$ . The next lemma shows that this always occurs when  $M$  is compact with even first Betti number.

LEMMA 4. *Let  $(c, J_1, J_2)$  be a bihermitian structure on a compact oriented 4-manifold  $M$  with even first Betti number. Then the standard metric  $g$  of  $(c, J_1)$  is also standard for  $(c, J_2)$  and the Lee forms  $\theta_1$  and  $\theta_2$  of  $J_1$  and  $J_2$  with respect to  $g$  satisfy*

$$\theta_1 + \theta_2 = 0. \quad (17)$$

*Proof.* Since  $M$  is compact, the equality (3) in Lemma 1 is equivalent to the fact that  $\theta_1 + \theta_2$  is closed. Let  $g$  be the standard metric of  $(J_1, c)$ , so that  $\delta\theta_1 = 0$

(see [10]). Let  $\beta$  be the harmonic part in the Hodge decomposition of  $\theta_1 + \theta_2$ , that is,  $\theta_1 + \theta_2 = \beta + d\sigma$ , where  $\sigma$  is a smooth function on  $M$ . Since  $b_1(M)$  is even, we have from [11] that  $\theta_1 = \delta\alpha$  for some 2-form  $\alpha$  on  $M$ . Integrating (4) we get

$$\begin{aligned} \int_M |\theta_1|^2 dV &= \int_M |\theta_2|^2 dV = \int_M |\beta + d\sigma - \delta\alpha|^2 dV \\ &= \int_M (|\theta_1|^2 + |d\sigma|^2 + |\beta|^2) dV. \end{aligned}$$

Hence  $\beta \equiv 0$  and  $d\sigma \equiv 0$ , that is,  $\theta_1 + \theta_2 = 0$ .

Putting together Lemmas 1, 3 and 4 we obtain the following.

**PROPOSITION 2.** *Let  $(c, J_1, J_2)$  be a bihermitian structure on a compact oriented 4-manifold  $M$ . Set  $\mathcal{D} = \{x \in M: J_1(x) = \pm J_2(x)\}$ . Then  $\mathcal{D}$  is a complex curve of both  $(M, J_1)$  and  $(M, J_2)$ . If moreover, the first Betti number of  $M$  is even, then each of  $(M, J_1)$  and  $(M, J_2)$  has an effective anti-canonical divisor.*

*Proof.* Since  $M$  is compact, we infer as above that  $\theta_1 + \theta_2$  is closed. Then Lemma 3 shows that  $\sigma_1$  is a holomorphic section of the holomorphic line bundle  $K_{J_1}^{-1} \otimes L$ , where  $L$  is the topologically trivial holomorphic line bundle, determined by the closed 1-form  $\theta_1 + \theta_2$ . Then, as the zero set of  $\sigma_1$ ,  $\mathcal{D}$  is a complex curve of  $(M, J_1)$ . If moreover, the first Betti number is even, we infer from Lemma 4 that the closed 1-form  $\theta_1 + \theta_2$  is exact. Hence  $L$  is the trivial holomorphic line bundle, and  $\sigma_1$  determines an anti-canonical divisor of  $(M, J_1)$ . The same argument still holds when considering  $J_2$  instead of  $J_1$ . This completes the proof of Proposition 2.

We are now ready to prove Theorem 1. Let  $g$  be the standard metric of  $c$  with respect to  $J_1$  and  $J_2$  (Lemma 4) and let  $\sigma_1$  be the  $(0, 2)$ -form with respect to  $J_1$  appearing in Lemma 3, viewed as a section of the anti-canonical bundle  $K_{J_1}^{-1}$ . By Lemmas 3 and 4,  $\sigma_1$  is a holomorphic section of  $K_{J_1}^{-1}$  and  $\mathcal{D}$  is the zero set of  $\sigma_1$ . Then, either  $\mathcal{D}$  is empty, and then  $(c, J_1, J_2)$  is a strongly bihermitian structure, or, by Proposition 2, the plurigenera of  $(M, J_1)$  all vanish, and then the Kodaira dimension of  $(M, J_1)$  is equal to  $-\infty$ . Reversing the roles of  $J_1$  and  $J_2$ , we see that the Kodaira dimension of  $(M, J_2)$  is then also equal to  $-\infty$ .

In the former case,  $\sigma_1$  is a non-vanishing holomorphic section of  $K_{J_1}^{-1}$ , so that  $(M, J_1)$  is either the 4-torus or the K3 surface, and so is  $(M, J_2)$  (cf. [6]).

In the latter case, the Kodaira classification shows that  $(M, J_1)$  (and also  $(M, J_2)$ ) is either a ruled surface of genus  $g \geq 1$  or a rational surface (see [6]). Moreover, since  $(M, J_1)$  has an effective anti-canonical divisor (see Proposition 2), the same is true for the minimal model of  $M$ . Note that a minimal ruled surface has a Kähler metric of negative total scalar curvature (and hence admits no holomorphic sections of the anti-canonical bundle) as soon as its genus is greater than 1. So, the minimal model of  $(M, J_1)$  is either  $\mathbb{C}P^2$  or a minimal ruled surface of genus at most 1. According to the ‘blowing down’ formula [6], every holomorphic section of the anti-canonical bundle of  $(M, J_1)$  is the pull-back of an anti-canonical section of the blown down surface, and vanishes at the blown up point; the same holds for  $(M, J_2)$ . This completes the proof of Theorem 1.

## 3. Bihermitian structures and symplectic forms

Recall that a holomorphic symplectic structure on a complex surface  $(M, J)$  is a nowhere-vanishing holomorphic (complex) 2-form  $\Omega$ . It is well known that the complex structure  $J$  is then determined by the real and the imaginary parts of the 2-form  $\Omega$ ; more precisely, we have the following.

LEMMA 5 [18, 14]. *Let  $M$  be an oriented 4-manifold and  $(\Phi_1, \Phi_2)$  be a pair of non-degenerate real 2-forms on  $M$  satisfying the conditions*

$$\Phi_1 \wedge \Phi_1 = \Phi_2 \wedge \Phi_2, \quad \Phi_1 \wedge \Phi_2 = 0. \quad (18)$$

*Then there is a unique almost-complex structure  $J$  on  $M$  such that the 2-form  $\Omega = \Phi_1 - i\Phi_2$  is of type  $(2, 0)$  with respect to  $J$ . If moreover  $\Phi_1$  and  $\Phi_2$  are closed, then  $J$  is integrable and  $\Omega$  defines a holomorphic symplectic structure on  $(M, J)$ .*

*Proof.* Since  $\Phi_1$  and  $\Phi_2$  are non-degenerate, we define an automorphism  $J$  on  $M$  by putting  $\Phi_2(\cdot, \cdot) = \Phi_1(J\cdot, \cdot)$ . The conditions (18) imply that  $J^2 = -\text{Id}$ , that is,  $J$  is an almost-complex structure and  $\Omega := \Phi_1 - i\Phi_2$  is a  $(2, 0)$ -form with respect to  $J$ ; moreover,  $J$  is clearly the only almost-complex structure such that  $\Omega$  is of type  $(2, 0)$  with respect to  $J$ . If  $\Phi_1$  and  $\Phi_2$  are closed, then  $J$  is integrable. Indeed, fix any conformal class  $c$ , compatible with  $J$ . Since  $\Omega$  is of type  $(2, 0)$  with respect to  $J$ , the forms  $\Phi_1$  and  $\Phi_2$  are both self-dual with respect to  $c$  and have the same, non-vanishing, norm, by (18). We can then choose a metric  $g$  in  $c$  such that  $\Phi_1$  and  $\Phi_2$  are the Kähler forms of two anti-commuting almost Kähler structures,  $J_1$  and  $J_2$ . Then, according to [20] we infer that  $J = J_1 \circ J_2$  is integrable.

*Proof of Theorem 2.* (i) Let  $M$  be either a 4-dimensional torus or a K3 surface and let  $(c, J_1, J_2)$  be a bihermitian structure on  $M$ . According to Theorem 1,  $(c, J_1, J_2)$  is a strongly bihermitian structure. It follows that the 2-forms  $\Phi_0, \Phi_1, \Phi_2$  defined in the previous section are well defined on  $M$ . Recall that  $\Phi_0, \Phi_1, \Phi_2$  are the Kähler forms of the almost-complex structures  $I_0, I_1, I_2$  defined by (5) with respect to some metric  $g$  in the conformal class  $c$ . If  $g$  is chosen to be the common standard metric of  $J_1$  and  $J_2$  (see Lemma 4), then Lemma 2 simply reads

$$\tau = d \ln q.$$

It follows that  $q^{-1}\Phi_0, q^{-1}\Phi_1, q^{-1}\Phi_2$  are harmonic self-dual 2-forms, or equivalently,  $\Omega_1 = q^{-1}(\Phi_0 - i\Phi_1)$  and  $\Omega_2 = q^{-1}(\Phi_0 - i\Phi_2)$  are holomorphic 2-forms with respect to  $J_1$  and  $J_2$  respectively. Condition (c) in Theorem 2 then follows from (7) and (8).

(ii) Consider the triple of 2-forms

$$\left\{ \Phi_0, \Phi_1, \Phi'_2 = \frac{1}{\sqrt{1-p^2}}(p\Phi_1 + \Phi_2) \right\}.$$

Any two of these three forms satisfy the conditions (18) of Lemma 5. We thus get three almost-complex structures, say  $I_0, I_1, J_1$ , corresponding to the pairs  $\{\Phi_1, \Phi'_2\}$ ,  $\{\Phi'_2, \Phi_0\}$  and  $\{\Phi_0, \Phi_1\}$  respectively, where  $J_1$  is integrable. It is easy to check that the almost-complex structures  $I_0, I_1, J_1$  pairwise anticommute, and

hence determine a conformal class,  $c$ , defined by decreeing that the frame  $\{X, I_0X, I_1X, J_1X\}$  is  $c$ -orthonormal for any non-vanishing vector field  $X$ . By the definition of  $c$ , the forms  $\Phi_0, \Phi_1, \Phi_2$  are  $c$ -self-dual, and so are the forms  $\bar{\Phi}_0, \bar{\Phi}_1, \bar{\Phi}_2$ . It follows that the *integrable* almost-complex structure,  $J_2$ , determined by the pair  $\{\Phi_0, \Phi_2\}$  via Lemma 5, is also compatible with  $c$ . We thus get a conformal bihermitian structure, actually a strongly bihermitian one since  $|p| < 1$ . The last statement is clear.

#### 4. Construction of bihermitian structures

With Theorem 2 to hand, we easily provide a number of examples of strongly bihermitian surfaces. First, we recover Kobak’s construction for  $\mathbb{R}^4$  and  $T^4$  as follows.

##### 4.1. Kobak’s examples on $\mathbb{R}^4$ and $T^4$

Let  $g_1$  be the flat metric of  $\mathbb{R}^4$  and let  $\{\omega_I, \omega_J, \omega_K\}$  and  $\{\omega_{\bar{I}}, \omega_{\bar{J}}, \omega_{\bar{K}}\}$  be the Kähler forms of the canonical flat hyperkähler structure on  $(\mathbb{C}^2, g_1)$  and  $(\overline{\mathbb{C}^2}, g_1)$  respectively:

$$\begin{aligned} \omega_I &= \operatorname{Re}(dz_1 \wedge dz_2), & \bar{\omega}_I &= \operatorname{Re}(dz_1 \wedge d\bar{z}_2), \\ \omega_K &= \operatorname{Im}(dz_1 \wedge dz_2), & \bar{\omega}_K &= \operatorname{Im}(dz_1 \wedge d\bar{z}_2), \\ \omega_J &= \frac{1}{2}i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2), & \bar{\omega}_J &= \frac{1}{2}i(dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2), \end{aligned}$$

where  $(z_1, z_2)$  are the canonical coordinates of  $\mathbb{C}^2$ .

Consider the triple

$$\Phi_0 = \omega_I, \quad \Phi_1 = \omega_K, \quad \Phi_2 = \omega_J + p(\omega_K - \bar{\omega}_J), \tag{19}$$

where  $p$  is a smooth (real-valued) function on  $\mathbb{C}^2$  satisfying  $|p| < 1$ . According to Theorem 2, the triple  $\{\Phi_0, \Phi_1, \Phi_2\}$  determines a (strongly) bihermitian structure on  $\mathbb{C}^2$  provided that  $\Phi_2$  is closed. This happens if and only if the function  $p$  is a solution of the following system:

$$\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)p = \left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right)p = 0, \tag{20}$$

where  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ .

If, moreover, the smooth function  $p$  is periodic with respect to all variables  $x_1, y_1, x_2, y_2$ , then  $\{\Phi_0, \Phi_1, \Phi_2\}$  induce a (strongly) bihermitian structure on the 4-dimensional torus  $T^4$ . It is easily seen that the corresponding conformal structure  $c$  is determined by the metric  $g_f = dz_1 \odot d\bar{z}_1 + f dz_2 \odot d\bar{z}_2$ , where  $f$  is the smooth positive function  $(1 - p)/(1 + p)$  (compare with [16]).

We may observe that, for some solutions  $p$  of (20), the corresponding bihermitian structures  $(c, J_1, J_2)$  admit an orientation-preserving isometry  $\phi$  such that  $\phi(J_1) = J_2$ . Indeed, let  $h(t)$  be an arbitrary smooth function on  $\mathbb{R}$ , satisfying  $|h| < 1$ . If we put  $p = h(y_1 - y_2)$ , then  $p$  is a solution of (20) and so determines a bihermitian conformal structure  $(c, J_1, J_2)$ . Let  $H(t)$  be a smooth function on  $\mathbb{R}$  satisfying  $dH(t)/dt = h(t)$  and consider the orientation-preserving diffeomorphism  $\phi$  of  $\mathbb{R}^4$  defined by

$$\phi(x_1, y_1, x_2, y_2) = (x_1, -y_2 + H(y_1 - y_2), x_2, y_1 - H(y_1 - y_2)).$$

It is easily checked that  $\phi(\Phi_0) = \Phi_0$ ,  $\phi(\Phi_1) = \Phi_2$  and  $\phi(\Phi_2) = -\Phi_1$ . In particular,  $\phi$  preserves the conformal structure  $c$  and satisfies  $\phi(J_1) = J_2$  and

$\phi(J_2) = -J_1$ . We also notice that with respect to the metric  $g$  in  $c$  which makes  $\Phi_0$ ,  $\Phi_1$  and  $\Phi_2$  self-dual forms of square-norm 2,  $\phi$  is an orientation-preserving isometry. If moreover  $H$  is a  $2\pi$ -periodic function (take, for example,  $H(t) = \frac{1}{2} \sin(t)$ ), we can perform the above construction on  $T^4 = \mathbb{R}^4 / 2\pi\mathbb{Z}^4$ .

#### 4.2 Strongly bihermitian structures as deformations of hyperkähler structures

The main idea in this section, consisting in deforming a hyperkähler structure by a Hamiltonian vector field in order to get a family of strongly bihermitian structures via Theorem 2, is due to D. Joyce (private communication).

We start our construction with *any* hyperkähler structure on a 4-dimensional manifold  $M$ , determined by a Riemannian metric  $g_0$ , a triple of pairwise anticommuting complex structures,  $K_0, K_1, K_2$ , and the corresponding triple of Kähler forms,  $\omega_0, \omega_1, \omega_2$ . Let  $\phi_t$  be a 1-parameter family of symplectomorphisms of the (real) symplectic manifold  $(M, \omega_0)$ , with  $\phi_0 = \text{Id}$ , and consider the triple of real symplectic 2-forms  $\Phi_0, \Phi_1, \Phi_2$  defined as follows:

$$\Phi_0 = \omega_0, \quad \Phi_1 = \omega_1, \quad \Phi_2 = \phi_t \omega_2. \quad (21)$$

We check easily that this triple satisfies the conditions (a), (b), (c) of Theorem 2 when  $t$  is small enough (the latter condition ensures that the function  $p_t$  defined by  $\Phi_1 \wedge \Phi_2 = -p_t \Phi_0 \wedge \Phi_0$  satisfies  $|p_t| < 1$ ). By Theorem 2, we thus obtain a strongly bihermitian structure  $(c, J_1, J_2)$ , where the complex structures  $J_1$  and  $J_2$  are related to the former ones,  $K_0, K_1, K_2$ , by

$$J_1 = -K_2, \quad J_2 = \phi_t^{-1} K_1, \quad (22)$$

whereas the angle function  $p_t$  is given by

$$p_t = -\frac{1}{4} \text{trace}(K_2 \circ (\phi_t^{-1} K_1)). \quad (23)$$

We consider the case when  $\phi_t = \exp(tV)$ , where  $V = K_0(\text{grad } f)$  is a Hamiltonian vector field with respect to  $\omega_0$  (here,  $f$  is a real function and  $\text{grad } f$  is the gradient of  $f$  with respect to the hyperkähler metric  $g_0$ ).

According to the last statement of Theorem 2, we have to show that  $f$  can be chosen so that the angle function  $p_t$  is non-constant for some  $t$ . This follows from the following lemma, where  $\Delta$  denotes the Laplacian with respect to  $g_0$ .

LEMMA 6. *The derivative of  $p_t$  with respect to  $t$  is given by*

$$\left. \frac{dp_t}{dt} \right|_{t=0} = \frac{1}{2} \Delta f. \quad (24)$$

*In particular, if  $\Delta f$  is not constant on  $M$ , then  $p_t$  is non-constant for infinitely many (small) values of  $t$ .*

*Proof.* Equation (24) is an easy consequence of the expression (23) of  $p_t$ . The last statement follows immediately.

PROPOSITION 3. (i) *Any hyperkähler structure on any K3 surface or any complex torus can be deformed as above into a non-ASD strongly bihermitian structure.*

(ii) *Any hyperhermitian structure on a Hopf surface can be deformed into a non-ASD strongly bihermitian structure.*

*Proof.* (i) It only remains to prove that the strongly bihermitian structure obtained above, when deforming any hyperkähler structure by any *non-constant* function  $f$ , is not anti-self-dual. This follows from the fact that any ASD bihermitian structure on a K3 surface or a complex torus is actually (conformally) hyperkähler; see [19].

(ii) Let  $M = (\mathbb{C}^2 - \{(0, 0)\})/\Gamma$  be any Hopf surface admitting a hyperhermitian structure. Then, the induced hyperhermitian structure on the universal covering  $(\mathbb{C}^2 - \{(0, 0)\})$  is conformal to a flat hyperkähler structure  $(g_0, K_0, K_1, K_2; \omega_0, \omega_1, \omega_2)$ . Choose any smooth function  $f$  on  $(\mathbb{C}^2 - \{(0, 0)\})$  that scales under  $\Gamma$  so that its symplectic gradient with respect to  $\omega_0$  is  $\Gamma$ -invariant (take any smooth function on  $M$ , viewed as a  $\Gamma$ -invariant function on  $(\mathbb{C}^2 - \{(0, 0)\})$ , and multiply it by the square of the distance to the origin). By deforming the flat hyperkähler structure by  $f$  as above, we get a (conformal) strongly bihermitian structure on  $(\mathbb{C}^2 - \{(0, 0)\})$ , which clearly descends to  $M$ . The latter is non-ASD as soon as  $f$  is chosen so that  $\Delta f$  is not constant, where  $\Delta$  is the Laplacian of the flat hyperkähler metric. Indeed, any ASD bihermitian structure on a Hopf surface is actually hyperhermitian and lifts to a (conformal) hyperkähler structure on  $(\mathbb{C}^2 - \{(0, 0)\})$ ; see [19].

#### 4.3. Proof of Proposition 1

By [8], each hyperhermitian structure on a compact complex surface belongs to one of the three types considered in Proposition 3. Then, Proposition 1 follows directly from the latter.

### 5. Complex surfaces with odd first Betti number

In this section, we consider the ‘odd case’, where the (connected) 4-manifold  $M$  is compact, with odd first Betti number, endowed with a bihermitian structure  $(c, J_1, J_2)$ , and we focus our attention on the (closed) 1-form  $\theta_1 + \theta_2$ , where  $\theta_1$  and  $\theta_2$  are the Lee forms of  $J_1$  and  $J_2$  with respect to some metric  $g$  in the conformal class  $c$ . Recall that this 1-form vanishes identically when  $M$  is of Kähler type and  $g$  is the (common) standard metric of  $J_1$  and  $J_2$ ; see Lemma 4. This is no longer true in general in the odd case. Indeed, consider the two types of ASD bihermitian structures on a Hopf surface  $S^1 \times S^3$  admitting two distinct hyperhermitian structures  $(c, \mathcal{F})$  and  $(c, \mathcal{F}')$  as in § 1. Then, if  $g$  is the standard metric of  $S^1 \times S^3$ , the common Lee form  $\theta'$  of  $(g, \mathcal{F}')$  is equal to  $-\theta$ , where  $\theta$  is the common Lee form of  $(g, \mathcal{F})$  (see [12]). It follows that, if  $J_1$  and  $J_2$  both belong to  $\mathcal{F}$ , we have  $\theta_1 + \theta_2 = 2\theta \neq 0$ ; whereas, if  $J_1$  and  $J_2$  belong to distinct families, we have  $\theta_1 + \theta_2 = 0$ .

As a matter of fact, the only examples known at present where the relation  $\theta_1 + \theta_2 = 0$  is satisfied for a bihermitian structure on a compact 4-manifold with odd first Betti number are *anti-self-dual*, namely when either  $(M, c)$  is of the latter type or when  $c$  is one of the anti-self dual metrics on  $M = (S^1 \times S^3) \# n \overline{\mathbb{C}P}^2$  constructed by C. LeBrun, and  $J_1$  and  $J_2$  are the corresponding compatible complex structures; see [19].

The following proposition shows that the relation  $\theta_1 + \theta_2 = 0$  never holds in the odd case for *strongly* bihermitian structures.

Let  $\mathcal{D}^+$  and  $\mathcal{D}^-$  be the complex curves (for both  $J_1$  and  $J_2$ ) defined by

$$\mathcal{D}^+ = p^{-1}(1) = \{x \in M: J_1(x) = J_2(x)\}$$

and

$$\mathcal{D}^- = p^{-1}(-1) = \{x \in M: J_1(x) = -J_2(x)\},$$

where, we recall,  $p = -\frac{1}{4} \text{trace}(J_1 \circ J_2)$  is the angle function. We then have the following.

**PROPOSITION 4.** *Let  $(M, c, J_1, J_2)$  be a compact bihermitian surface and suppose that there is a metric  $g$  in the conformal class  $c$  such that  $\theta_1 + \theta_2 = 0$ . Then the first Betti number of  $M$  is odd if and only if the complex curves  $\mathcal{D}^+$  and  $\mathcal{D}^-$  are both non-empty.*

*Proof.* Denote by  $\mathcal{U}^+$  and  $\mathcal{U}^-$  the open sets where  $J_1 \neq J_2$  and  $J_1 \neq -J_2$  respectively. Keeping the notation of §2, we consider the self-dual 2-forms

$$\omega_+(\cdot, \cdot) = F_1(\cdot, \cdot) - \frac{1}{1-p} \Phi(J_1 \cdot, \cdot)$$

and

$$\omega_-(\cdot, \cdot) = F_1(\cdot, \cdot) + \frac{1}{1+p} \Phi(J_1 \cdot, \cdot),$$

defined on  $\mathcal{U}^+$  and  $\mathcal{U}^-$  respectively. By using formulae appearing in the proof of Lemma 2, one can easily check that we have  $\delta\omega_+ = -\frac{1}{2}\omega_+(\theta_1 + \theta_2)$  and  $\delta\omega_- = -\frac{1}{2}\omega_-(\theta_1 + \theta_2)$  on  $\mathcal{U}^+ \cap \mathcal{U}^-$ . According to [19, Proposition 1.3],  $\mathcal{U}^+ \cap \mathcal{U}^-$  is a dense subset of  $\mathcal{U}^+$  and  $\mathcal{U}^-$ , so that  $\omega_+$  and  $\omega_-$  are harmonic self-dual 2-forms with respect to the chosen metric  $g$  on  $\mathcal{U}^+$  and  $\mathcal{U}^-$  respectively. Suppose now that  $\mathcal{D}^+$  is empty, that is,  $\mathcal{U}^+ = M$ . Then the harmonic self-dual 2-form  $\omega_+$  has non-zero trace, and hence the first Betti number of  $M$  is even by [11, Proposition II.4].

Conversely, suppose that the first Betti number is even and let  $\mathcal{D}^+$  be non-empty. We have to prove that  $\mathcal{D}^-$  is empty. Let  $g$  be the standard metric of  $c$  and consider a harmonic self-dual form  $\omega$  of trace equal to 1 with respect to  $(g, J_1)$  [11, Proposition II.4], that is,  $\omega = F_1 + \text{Re}(\alpha_1)$ , where  $F_1$  is the Kähler form of  $(g, J_1)$  and  $\alpha_1$  is a section of  $\Lambda_{J_1}^{0,2}M$ . Since the metric  $g$  is also standard for  $(c, J_2)$  (Lemma 4), we have  $\omega = \lambda F_2 + \text{Re}(\alpha_2)$ , where  $\lambda$  is a constant,  $F_2$  is the Kähler form of  $(g, J_2)$  and  $\alpha_2$  is a section of  $\Lambda_{J_2}^{0,2}M$ . We claim that  $\lambda = 1$ . Indeed, for any point  $x \in \mathcal{D}^+$  we have

$$\omega(x) = F_1(x) + \text{Re}(\alpha_1(x)) = \lambda F_1(x) + \text{Re}(\alpha_2(x));$$

hence  $\lambda = 1$ , that is,  $F_1 + \text{Re}(\alpha_1) = F_2 + \text{Re}(\alpha_2)$ . The latter equality shows that there are no points where  $J_1 = -J_2$ , that is,  $\mathcal{D}^-$  is empty.

**COROLLARY 2.** *Let  $(M, c, J_1, J_2)$  be a compact bihermitian surface with odd first Betti number and assume that the conformal class  $c$  contains a metric  $g$  such that  $\theta_1 + \theta_2 = 0$ . Then  $(c, J_1, J_2)$  is not strongly bihermitian and  $(M, J_1)$  (equivalently  $(M, J_2)$ ) is obtained from either a Hopf surface or a parabolic Inoue surface by blowing up points along a unique smooth elliptic curve. In particular,  $M$  is diffeomorphic to  $(S^1 \times S^3) \# k \overline{\mathbb{C}P^2}$  for some  $k \geq 0$ .*

*Proof.* The proof of Proposition 1 shows that if a bihermitian surface  $(M, c, J_1, J_2)$  with odd first Betti number satisfies  $\theta_1 + \theta_2 = 0$  for some metric  $g$

in  $c$ , then either  $(M, J_1)$  (equivalently  $(M, J_2)$ ) has a trivial (anti-)canonical bundle (that is,  $(M, J_1)$  is a primary Kodaira surface, see [6]), or else it has an effective anti-canonical divisor, so that  $(M, J_1)$  belongs to Class VII in Kodaira’s classification. It is easily seen that a primary Kodaira surface (and hence any Kodaira surface)  $M$  admits no bihermitian structures at all. Indeed, let  $(c, J_1, J_2)$  be such a structure. Since the Kodaira dimension is an oriented smooth invariant (see [15]), both  $(M, J_1)$  and  $(M, J_2)$  are primary Kodaira surfaces, and hence have trivial canonical bundles. There thus exist non-vanishing  $J_1$ - and  $J_2$ -holomorphic 2-forms,  $\Omega_1$  and  $\Omega_2$ . Since  $J_1$  and  $J_2$  are both compatible with the conformal structure  $c$ , the real and the imaginary parts of  $\Omega_1$  and  $\Omega_2$  define four real harmonic self-dual 2-forms (with respect to  $c$ ). Since  $J_1$  and  $J_2$  are independent at some point, they span a subspace of  $H^2(M, \mathbb{R})$  of dimension at least 3. This contradicts the fact that  $b_2^+(M) = 2$ . It follows that both  $(M, J_1)$  and  $(M, J_2)$  belong to Class VII and each has an effective anti-canonical divisor. Moreover, by Proposition 4,  $\mathcal{D}^+$  and  $\mathcal{D}^-$  are non-empty. We then conclude as in [19] (see, in particular, the proofs of Proposition 3.11, Corollary 3.12, Corollary 3.14, Theorem 4.1, Proposition 5.1 and Theorem 5.2 in [19]).

6. The angle-function of a bihermitian structure

LEMMA 7. *Let  $(M, c, J_1, J_2)$  be a bihermitian surface. Let  $p$  be the angle-function defined by  $p = -\frac{1}{4} \text{trace}(J_1 \circ J_2)$ . Then, the differential  $dp$  and the Laplacian  $\Delta p$  with respect to some metric  $g$  in  $c$  are given by*

$$dp = \frac{1}{4}[J_1, J_2](\theta_1 - \theta_2) \tag{25}$$

and

$$\Delta p = -\frac{1}{2}g(dp, \theta_1 + \theta_2) + \frac{1}{2}p|\theta_1 - \theta_2|^2 + g([J_1, J_2], d\theta_1), \tag{26}$$

where, we recall,  $[J_1, J_2] = J_1 \circ J_2 - J_2 \circ J_1$ .

*Proof.* We first observe that  $p = \frac{1}{2}g(J_1, J_2)$ , for any metric  $g$  in  $c$ , so that  $dp(X) = \frac{1}{2}(g(\nabla_X J_1, J_2) + g(J_1, \nabla_X J_2))$  for any vector field  $X$ . Since  $J_1$  and  $J_2$  are both integrable, we have

$$(\nabla_X J_1)(Y) = \frac{1}{2}(g(X, Y)J_1\theta_1 + F_1(X, Y)\theta_1 + \theta_1(J_1Y)X - \theta_1(Y)J_1X), \tag{27}$$

and

$$(\nabla_X J_2)(Y) = \frac{1}{2}(g(X, Y)J_2\theta_2 + F_2(X, Y)\theta_2 + \theta_2(J_2Y)X - \theta_2(Y)J_2X), \tag{28}$$

where  $F_1$  and  $F_2$  are the Kähler forms of  $(g, J_1)$  and  $(g, J_2)$  respectively. Now we easily compute

$$g((\nabla_X J_1), J_2) + g(J_1, (\nabla_X J_2)) = \frac{1}{2}g([J_1, J_2](\theta_1 - \theta_2), X),$$

and (25) follows. To obtain (26), we have to calculate  $\delta([J_1, J_2](\theta_1 - \theta_2))$ . Identifying 2-forms and skew-symmetric endomorphisms by  $g$ , we have

$$\delta([J_1, J_2](\theta_1 - \theta_2)) = -(\delta[J_1, J_2])(\theta_1 - \theta_2) + 2g([J_1, J_2], d(\theta_1 - \theta_2)). \tag{29}$$

By using (27), (28) and (2), we get

$$\begin{aligned}\delta(J_1 \circ J_2) &= J_1(\delta J_2) - \sum_{i=1}^4 (\nabla_{e_i} J_1)(J_2 e_i) \\ &= -(J_1 \circ J_2)\theta_2 - \frac{1}{2}(4p\theta_1 + (J_1 \circ J_2 + J_2 \circ J_1)(\theta_1)) \\ &= -(J_1 \circ J_2)\theta_2 - p\theta_1.\end{aligned}$$

Hence

$$\begin{aligned}\delta[J_1, J_2] &= (J_2 \circ J_1)\theta_1 - (J_1 \circ J_2)\theta_2 - p(\theta_1 - \theta_2) \\ &= -\frac{1}{2}[J_1, J_2](\theta_1 + \theta_2) - 2p(\theta_1 - \theta_2).\end{aligned}\tag{30}$$

By using (3) and substituting (30) into (29) we eventually obtain (26).

As a consequence of the above lemma we get the following.

**COROLLARY 3.** *Let  $(M, c, J_1, J_2)$  be a compact strongly bihermitian surface. Then the positive Weyl tensor vanishes at some point of  $M$ .*

*Proof.* It follows from [3, Lemma 2 and (13)] that at any point where  $W^+$  does not vanish we have

$$\begin{aligned}J_2 \circ J_1 &= \varepsilon \left[ \frac{3\lambda_0}{(\lambda_+ - \lambda_-)} \text{Id} + \frac{1}{(\lambda_+ - \lambda_-)} (d\theta_1)_+ \right], \\ J_1 \circ J_2 &= \varepsilon \left[ \frac{3\lambda_0}{(\lambda_+ - \lambda_-)} \text{Id} + \frac{1}{(\lambda_+ - \lambda_-)} (d\theta_2)_+ \right],\end{aligned}$$

where  $\lambda_+ \geq \lambda_0 \geq \lambda_-$  are the eigenvalues of  $W^+$  and the sign  $\varepsilon \in \{-1, 1\}$  depends on the point. We thus obtain

$$(d\theta_1)_+ = \varepsilon \frac{(\lambda_+ - \lambda_-)}{2} [J_1, J_2],\tag{31}$$

which clearly holds everywhere on  $M$ . In general, the sign  $\varepsilon$  in (31) is not constant, but in the case when  $(c, J_1, J_2)$  is strongly bihermitian,  $W^+$  is degenerate if and only if it vanishes. Assume, for a contradiction, that  $W^+$  is everywhere non-degenerate. Then, the right-hand side of (31) does not vanish and  $\varepsilon$  can be chosen equal to  $+1$  or  $-1$  everywhere, in fact equal to  $1$  by replacing  $J_1$  by  $-J_1$  if necessary. It then follows that (26) can be written as

$$\Delta p = -\frac{1}{2}g(dp, \theta_1 + \theta_2) + \frac{1}{2}p|\theta_1 - \theta_2|^2 + \frac{1}{2}(\lambda_+ - \lambda_-)|[J_1, J_2]|^2.\tag{32}$$

Let  $x_0 \in M$  be a point where the angle function  $p$  achieves its minimum. We thus have  $dp(x_0) = 0$  and  $\Delta p(x_0) \leq 0$ . Since  $(c, J_1, J_2)$  is strongly bihermitian,  $[J_1, J_2]$  is everywhere non-zero, and hence invertible. We then infer from (25) that  $\theta_1$  and  $\theta_2$  are equal at  $x_0$ , so that

$$(\lambda_+ - \lambda_-)|[J_1, J_2]|^2 \leq 0$$

at  $x_0$ , by (32). This contradicts the fact that  $|[J_1, J_2]|$  and  $\lambda_+ - \lambda_-$  are both (strictly) positive.

*References*

1. D. V. ALEKSEEVSKY and M. M. GRAEV, 'Calabi–Yau metric on the Fermat surface. Isometries and totally geodesics submanifolds', *J. Geom. Phys.* 7 (1990) 21–43.
2. D. V. ALEKSEEVSKY, S. MARCHIAFAVA and M. PONTECORVO, 'Compatible complex structures on almost quaternionic manifolds', *Trans. Amer. Math. Soc.* 351 (1999) 997–1014.
3. V. APOSTOLOV and P. GAUDUCHON, 'The Riemannian Goldberg–Sachs Theorem', *Internat. J. Math.* 8 (1997) 421–439.
4. V. APOSTOLOV and T. DRAGHICI, 'Hermitian conformal classes and almost Kähler structures on 4-manifolds', *Differential Geom. Appl.* to appear.
5. M. F. ATIYAH, N. J. HITCHIN and I. M. SINGER, 'Self-duality in 4-dimensional Riemannian geometry', *Proc. Roy. Soc. London Ser. A* 362 (1978) 425–461.
6. W. BARTH, C. PETERS and A. VAN DE VEN, *Compact complex surfaces* (Springer, Heidelberg, 1984).
7. C. P. BOYER, 'Conformal duality and compact complex surfaces', *Math. Ann.* 274 (1986) 517–526.
8. C. P. BOYER, 'A note on hyperhermitian 4-manifolds', *Proc. Amer. Math. Soc.* 102 (1988) 157–164.
9. A. DERDZIŃSKI, 'Self-dual Kähler manifolds and Einstein manifolds of dimension four', *Compositio Math.* 49 (1983) 405–433.
10. P. GAUDUCHON, 'Le théorème de l'excentricité nulle', *C. R. Acad. Sci. Paris Ser. A* 285 (1977) 387–390.
11. P. GAUDUCHON, 'La 1-forme de torsion d'une variété hermitienne compacte', *Math. Ann.* 267 (1984) 495–518.
12. P. GAUDUCHON, 'Structures de Weyl–Einstein, espaces de twisteurs et variétés de type  $S^1 \times S^3$ ', *J. reine angew. Math.* 469 (1995) 1–50.
13. P. GAUDUCHON, 'Canonical connections for almost-hypercomplex structures', *Complex analysis and geometry* (ed. V. Ancona, E. Ballico, R. M. Mirò-Roig and A. Silva), Pitman Research Notes in Mathematics Series 366 (Addison Wesley Longman, Harlow, 1997) 123–136.
14. H. GEIGES and J. GONZALO, 'Contact geometry and complex surfaces', *Invent. Math.* 121 (1995) 147–209.
15. R. FRIEDMAN and Z. QIN, 'The smooth invariance of the Kodaira dimension of a complex surface', *Math. Res. Lett.* 1 (1994) 369–376.
16. P. KOBAK, 'Explicit doubly-Hermitian metrics', *Differential Geom. Appl.* 10 (1999) 179–185.
17. A. LICHTNEROWICZ, *Géométrie des groupes de transformations* (Dunod, Paris, 1958).
18. V. V. LYCHAGIN, V. N. RUBTSOV and I. V. CHEKALOV, 'A classification of Monge–Ampère equations', *Ann. Sci. École Norm. Sup.* (4) 26 (1993) 281–308.
19. M. PONTECORVO, 'Complex structures on Riemannian 4-manifolds', *Math. Ann.* 309 (1997) 159–177.
20. S. SALAMON, 'Special structures on 4-manifolds', *Riv. Mat. Univ. Parma.* (4) 17 (1991) 109–123.
21. I. VAISMAN, 'Some curvature properties of complex surfaces', *Ann. Mat. Pura Appl.* 32 (1982) 1–18.

V. Apostolov  
*Mathematical Institute*  
 24–28 St Giles'  
 Oxford OX1 3LB  
 E-mail: apostolo@maths.ox.ac.uk

P. Gauduchon  
*Centre de Mathématiques*  
 UMR 7640 du CNRS  
 École Polytechnique  
 91128 Palaiseau Cedex  
 France  
 E-mail: pg@math.polytechnique.fr

G. Grantcharov  
*Department of Mathematics*  
*University of California at Riverside*  
 Riverside  
 CA 52521  
 U.S.A.  
 E-mail: geogran@math.ucr.edu