Examples of Connections

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1 Preliminaries

Consider a rank r complex vector bundle $\pi : E \to M$. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of M such that

$$E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^r$$

for every $U_\alpha \in \mathcal{U}$, and let $e^1_\alpha, \dots, e^r_\alpha$ be trivializing sections. We have maps

$$g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL(r, \mathbb{C})$$

where $U_{\alpha\beta} = U_\alpha \cap U_\beta$ for $U_\alpha, U_\beta \in \mathcal{U}$.These tell us how to move between two trivializing sections on the overlap $U_{\alpha\beta}$. In particular, we have

$$e^i_\alpha = g_{\alpha\beta}{}^i_j e^j_\beta.$$

A connection d_A is a local opperator, determined by its restriction to $\mathcal{U}$. The space of connections on the bundle E is denoted $\mathcal{A}(E)$, and for $A \in \mathcal{A}(E|_{U_\alpha})$, $B \in \mathcal{A}(E|_{U_\beta})$ we have

$$\begin{aligned}
d_A e^i_\alpha &= d_A(g_{\alpha\beta}{}^i_j e^j_\beta) \\
A^i_j e^j_\alpha &= dg_{\alpha\beta}{}^i_j e^j_\beta + g_{\alpha\beta}{}^i_j B^j_k e^k_\beta \\
A^i_j e^j_\alpha &= dg_{\alpha\beta}{}^i_j g_{\beta\alpha}{}^j_k e^k_\alpha + g_{\alpha\beta}{}^i_j B^j_k g_{\beta\alpha}{}^k_l e^l_\alpha
\end{aligned}$$

where $g_{\beta\alpha} = g^{-1}_{\alpha\beta}$. Now, renaming indicies,

$$\begin{aligned}
A^i_n e^n_\alpha &= dg_{\alpha\beta}{}^i_j g_{\beta\alpha}{}^j_n e^n_\alpha + g_{\alpha\beta}{}^i_j B^j_k g_{\beta\alpha}{}^k_n e^n_\alpha \\
A^i_n e^n_\alpha &= (dg_{\alpha\beta}{}^i_j g_{\beta\alpha}{}^j_n + g_{\alpha\beta}{}^i_j B^j_k g_{\beta\alpha}{}^k_n) e^n_\alpha
\end{aligned}$$

so that in general,

$$A = (dg)g^{-1} + gBg^{-1}$$

as matricies.

2 The case of $\mathbb{CP}^1$

As a simple example, consider $T\mathbb{CP}^1 \to \mathbb{CP}^1$. Here our manifold M is

$$\mathbb{CP}^1 = (\mathbb{C}^2 \setminus \{0\}) \big/ _{(x_0:x_1)\sim(\lambda x_0:\lambda x_1)}$$

where $\lambda \neq 0$ and E is

$$T\mathbb{CP}^1 = \{(x,v) : x \in \mathbb{CP}^1, v \in T_x\mathbb{CP}^1\}.$$

Construct $\mathcal{U} = \{U_0, U_1\}$ by taking

$$U_0 = \{(x_0 : x_1) \in \mathbb{CP}^1 : x_0 \neq 0\} = \left\{\left(1 : \frac{x_1}{x_0}\right)\right\} = \{z \in \mathbb{C}\} \subset \mathbb{CP}^1$$

$$U_1 = \{(x_0 : x_1) \in \mathbb{CP}^1 : x_1 \neq 0\} = \left\{\left(\frac{x_0}{x_1} : 1\right)\right\} = \{w \in \mathbb{C}\} \subset \mathbb{CP}^1$$

so that each of the U_α is a copy of $\mathbb{C}$ and

$$\begin{array}{ccc} U_0 & \longrightarrow & U_1 \\ z & \longmapsto & \frac{1}{z} = w. \end{array}$$

Now $T\mathbb{CP}^1|_{U_\alpha}$ is a copy of $\mathbb{C} \times \mathbb{C}$ so that

$$T\mathbb{CP}^1|_{U_\alpha} \cong U_\alpha \times \mathbb{C}$$

via the maps

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C} & \longleftrightarrow & U_0 \times \mathbb{C} \\ \left(z, \frac{\partial}{\partial z}\right) & \longmapsto & (z, 1) \end{array}$$

and

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C} & \longleftrightarrow & U_1 \times \mathbb{C} \\ \left(w, \frac{\partial}{\partial w}\right) & \longmapsto & (w, 1). \end{array}$$

The trivializing sections are $e_0 = \frac{\partial}{\partial z}$ and $e_1 = \frac{\partial}{\partial w}$ and the transition function

$$g_{01} : U_{01} \longrightarrow GL(1, \mathbb{C})$$

is given by

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial w}{\partial z}\frac{\partial}{\partial w} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial z}\frac{1}{z}\frac{\partial}{\partial w} \\ \frac{\partial}{\partial z} &= -\frac{1}{z^2}\frac{\partial}{\partial w} \\ e_0 &= -w^2 e_1 \end{aligned}$$

thus $g_{01} = g = -w^2$ and $g_{10} = g^{-1} = \frac{-1}{w^2}$. In this case, we have only 1×1 matricies (functions) to worry about. If $A \in \mathcal{A}(T\mathbb{CP}^1|_{U_0})$ and $B \in \mathcal{A}(T\mathbb{CP}^1|_{U_1})$ then

$$\begin{aligned} A &= (dg)g^{-1} + gBg^{-1} \\ A &= (-2wdw)\left(-\frac{1}{w^2}\right) + B \\ A &= \frac{2}{w}dw + B. \qquad \bigstar \end{aligned}$$

Let

$$B = -\frac{2\bar{w}dw}{1 + w\bar{w}}.$$

Notice that B is defined on all of $\mathbb{C}$ (i.e all of U_1). We need to see that A is also well defined on $\mathbb{C}$ (i.e all of U_0), and that both functions agree on the overlap $U_{01} = \mathbb{C}^*$ (as in $\bigstar$).

$$\begin{aligned} A &= \frac{2}{w}dw - \frac{2\bar{w}}{1 + w\bar{w}}dw \\ &= \frac{2 + 2w\bar{w} - 2w\bar{w}}{w(1 + w\bar{w})}dw \\ &= \frac{2}{w(1 + w\bar{w})}dw \end{aligned}$$

Now $w = \frac{1}{z}$ and $dw = \frac{dw}{dz}dz = -\frac{1}{z^2}dz$ so that

$$\begin{aligned} &= \frac{2z^2\bar{z}}{1 + z\bar{z}}(-\frac{1}{z^2})dz \\ &= -\frac{2\bar{z}}{1 + z\bar{z}}dz \end{aligned}$$

and A is well defined on $\mathbb{C}$ as required. The curvature of A is defined

$$F_A = dA + A \wedge A = dA$$

since A is a 1×1 matrix.

$$\begin{aligned} F_A &= \frac{\partial A}{\partial \bar{z}}d\bar{z} \\ &= \frac{-2(1 + z\bar{z}) + 2\bar{z}(z)}{(1 + z\bar{z})^2}d\bar{z}dz \\ &= \frac{2}{(1 + z\bar{z})^2}dzd\bar{z} \end{aligned}$$

We want to see that

$$\chi(\mathbb{CP}^1) = \int_{\mathbb{CP}^1} Tr\left(\frac{iF_A}{2\pi}\right) = 2.$$

First, notice that $Tr(F_A) = F_A$ since F_A is a function in this case. Then, by removing a set of measure zero,

$$\int_{\mathbb{CP}^1} Tr\left(\frac{iF_A}{2\pi}\right) = \int_{\mathbb{CP}^1} \frac{iF_A}{2\pi} = \frac{i}{2\pi}\int_{\mathbb{C}} F_A.$$

Finally,

$$\begin{aligned}
\frac{i}{2\pi}\int_{\mathbb{C}} F_A &= \frac{i}{2\pi}\int_{\mathbb{C}} \frac{2}{(1+z\bar{z})^2} dz d\bar{z} \\
&= \frac{i}{\pi}\int_{\mathbb{R}^2} \frac{1}{(1+x^2+y^2)^2}(dx+idy)(dx-idy) \\
&= \frac{i}{\pi}\int_{\mathbb{R}^2} \frac{-2i}{(1+x^2+y^2)^2} dxdy \\
&= \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{2}{(1+x^2+y^2)^2} dxdy \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} \frac{2r}{(1+r^2)^2} drd\theta \\
&= 2\int_0^{\infty} \frac{d(1+r^2)}{(1+r^2)^2} \\
&= 2\left[-\frac{1}{1+r^2}\right]_0^{\infty} \\
&= 2
\end{aligned}$$

as required.

3 The case of $\mathbb{CP}^2$

Now consider the bundle $T\mathbb{CP}^2 \to \mathbb{CP}^2$. As before,

$$\mathbb{CP}^2 = (\mathbb{C}^3 \setminus \{0\}) \big/ (x_0 : x_1 : x_2) \sim (\lambda x_0 : \lambda x_1 : \lambda x_2)$$

for $\lambda \neq 0$ and

$$T\mathbb{CP}^2 = \{(x, v) : x \in \mathbb{CP}^2, v \in T_x\mathbb{CP}^2\}.$$

In this case, the cover $\mathcal{U}$ has three components,

$$U_0 = \{(x_0 : x_1 : x_2) \in \mathbb{CP}^2 : x_0 \neq 0\} = \left\{\left(1 : \frac{x_1}{x_0} : \frac{x_2}{x_0}\right)\right\} = \{(z_1, z_2) \in \mathbb{C}^2\} \subset \mathbb{CP}^2$$

$$U_1 = \{(x_0 : x_1 : x_2) \in \mathbb{CP}^2 : x_1 \neq 0\} = \left\{\left(\frac{x_0}{x_1} : 1 : \frac{x_2}{x_1}\right)\right\} = \{(w_1, w_2) \in \mathbb{C}^2\} \subset \mathbb{CP}^2$$

$$U_2 = \{(x_0 : x_1 : x_2) \in \mathbb{CP}^2 : x_2 \neq 0\} = \left\{\left(\frac{x_0}{x_2} : \frac{x_1}{x_2} : 1\right)\right\} = \{(y_1, y_2) \in \mathbb{C}^2\} \subset \mathbb{CP}^2$$

with maps

$$\begin{array}{rcl} U_0 & \longrightarrow & U_1 \\ (z_1, z_2) & \longmapsto & \left(\frac{1}{z_1}, \frac{z_2}{z_1}\right) = (w_1, w_2) \end{array}$$

and

$$\begin{array}{rcl} U_2 & \longrightarrow & U_1 \\ (y_1, y_2) & \longmapsto & \left(\frac{y_1}{y_2}, \frac{1}{y_2}\right) = (w_1, w_2). \end{array}$$

Each $T\mathbb{CP}^2|_{U_\alpha}$ is a copy of $\mathbb{C}^2 \times \mathbb{C}^2$ and

$$T\mathbb{CP}^2|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^2$$

via the maps

$$\begin{array}{rcl} \mathbb{C}^2 \times \mathbb{C}^2 & \longleftrightarrow & U_0 \times \mathbb{C}^2 \\ \left(z_1, z_2, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) & \longmapsto & (z_1, z_2, 1, 1) \end{array}$$

and

$$\begin{array}{rcl} \mathbb{C}^2 \times \mathbb{C}^2 & \longleftrightarrow & U_1 \times \mathbb{C}^2 \\ \left(w_1, w_2, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}\right) & \longmapsto & (w_1, w_2, 1, 1) \end{array}$$

and

$$\begin{array}{rcl} \mathbb{C}^2 \times \mathbb{C}^2 & \longleftrightarrow & U_2 \times \mathbb{C}^2 \\ \left(y_1, y_2, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right) & \longmapsto & (y_1, y_2, 1, 1). \end{array}$$

In this case, our trivialising sections are

$$\begin{array}{rcl} (e_0^1, e_0^2) = \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) & \text{on} & U_0 \\ (e_1^1, e_1^2) = \left(\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}\right) & \text{on} & U_1 \\ (e_2^1, e_2^2) = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right) & \text{on} & U_2. \end{array}$$

We need to build two transition functions

$$\begin{array}{rcl} g_{01} : U_{01} & \longrightarrow & GL(2, \mathbb{C}) \\ g_{21} : U_{21} & \longrightarrow & GL(2, \mathbb{C}) \end{array}$$

so that $e_0^i = g_{01}{}^i_j e_1^j$ and $e_2^i = g_{21}{}^i_j e_1^j$. To do this, write

$$\begin{aligned}
\begin{bmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_2} \end{bmatrix} &= \begin{bmatrix} \frac{\partial w_1}{\partial z_1} & \frac{\partial w_2}{\partial z_1} \\ \frac{\partial w_1}{\partial z_2} & \frac{\partial w_2}{\partial z_2} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial w_1} \\ \frac{\partial}{\partial w_2} \end{bmatrix} \\
[e_0^i] &= \begin{bmatrix} -\frac{1}{z_1^2} & -\frac{z_2}{z_1^2} \\ 0 & \frac{1}{z_1} \end{bmatrix} [e_1^i] \\
[e_0^i] &= \begin{bmatrix} -w_1^2 & -w_1 w_2 \\ 0 & w_1 \end{bmatrix} [e_1^i]
\end{aligned}$$

so that

$$\begin{aligned}
g_{01} &= \begin{bmatrix} -w_1^2 & -w_1 w_2 \\ 0 & w_1 \end{bmatrix} \\
g_{10} = g_{01}{}^{-1} &= -\frac{1}{w_1^3} \begin{bmatrix} w_1 & w_1 w_2 \\ 0 & -w_1^2 \end{bmatrix} \\
dg_{01} &= \begin{bmatrix} -2dw_1 & -(w_2 dw_1 + w_1 dw_2) \\ 0 & dw_1 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{bmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \end{bmatrix} &= \begin{bmatrix} \frac{\partial w_1}{\partial y_1} & \frac{\partial w_2}{\partial y_1} \\ \frac{\partial w_1}{\partial y_2} & \frac{\partial w_2}{\partial y_2} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial w_1} \\ \frac{\partial}{\partial w_2} \end{bmatrix} \\
[e_2^i] &= \begin{bmatrix} \frac{1}{y_2} & 0 \\ -\frac{y_1}{y_2^2} & -\frac{1}{y_2^2} \end{bmatrix} [e_1^i] \\
[e_2^i] &= \begin{bmatrix} w_2 & 0 \\ -w_1 w_2 & -w_2^2 \end{bmatrix} [e_1^i]
\end{aligned}$$

so that

$$\begin{aligned}
g_{21} &= \begin{bmatrix} w_2 & 0 \\ -w_1 w_2 & -w_2^2 \end{bmatrix} \\
g_{12} = g_{21}{}^{-1} &= -\frac{1}{w_2^3} \begin{bmatrix} -w_2^2 & 0 \\ w_1 w_2 & w_2 \end{bmatrix} \\
dg_{21} &= \begin{bmatrix} dw_2 & 0 \\ -(w_2 dw_1 + w_1 dw_2) & -2w_2 dw_2 \end{bmatrix}.
\end{aligned}$$

The aim now is to choose a local connection matrix $A \in \mathcal{A}(T\mathbb{CP}^2|_{U_1})$ and check that we get some local connection matricies $B \in \mathcal{A}(T\mathbb{CP}^2|_{U_0})$ and $C \in \mathcal{A}(T\mathbb{CP}^2|_{U_2})$, via

$$\begin{aligned}
B &= (dg_{01})g_{10} + g_{01} A g_{10} \\
C &= (dg_{21})g_{12} + g_{21} B g_{12}
\end{aligned}$$

that agree on the respective overlaps. First, compute the coefficients of B in terms of the coefficients of A:

$$
\begin{aligned}
&(dg_{01})g_{10} \\
&= -\frac{1}{w_1^3}\begin{bmatrix} -2dw_1 & -(w_2dw_1 + w_1dw_2) \\ 0 & dw_1 \end{bmatrix}\begin{bmatrix} w_1 & w_1w_2 \\ 0 & -w_1^2 \end{bmatrix} \\
&= -\frac{1}{w_1^3}\begin{bmatrix} -2w_1^2dw_1 & -2w_1^2w_2dw_1 + w_1^2(w_2dw_1 + w_1dw_2) \\ 0 & -w_1^2dw_1 \end{bmatrix} \\
&= -\frac{1}{w_1^3}\begin{bmatrix} -2w_1^2dw_1 & -w_1^2w_2dw_1 + w_1^3dw_2 \\ 0 & -w_1^2dw_1 \end{bmatrix}
\end{aligned}
$$

$$
(dg_{01})g_{10} = \begin{bmatrix} \frac{2}{w_1}dw_1 & \frac{w_2}{w_1}dw_1 - dw_2 \\ 0 & \frac{1}{w_1}dw_1 \end{bmatrix}
$$

$$
\begin{aligned}
&g_{01}Ag_{10} \\
&= -\frac{1}{w_1^3}\begin{bmatrix} -w_1^2 & -w_1w_2 \\ 0 & w_1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} w_1 & w_1w_2 \\ 0 & -w_1^2 \end{bmatrix} \\
&= -\frac{1}{w_1^3}\begin{bmatrix} -a_{11}w_1^2 - a_{21}w_1w_2 & -a_{12}w_1^2 - a_{22}w_1w_2 \\ a_{21}w_1 & a_{22}w_1 \end{bmatrix}\begin{bmatrix} w_1 & w_1w_2 \\ 0 & -w_1^2 \end{bmatrix} \\
&= -\frac{1}{w_1^3}\begin{bmatrix} -a_{11}w_1^3 - a_{21}w_1^2w_2 & -a_{11}w_1^3w_2 - a_{21}w_1^2w_2^2 + a_{12}w_1^4 + a_{22}w_1^3w_2 \\ a_{21}w_1^2 & a_{21}w_1^2w_2 - a_{22}w_1^3 \end{bmatrix}
\end{aligned}
$$

$$
g_{01}Ag_{10} = \begin{bmatrix} a_{11} + a_{21}\frac{w_2}{w_1} & a_{11}w_2 + a_{21}\frac{w_2^2}{w_1} - a_{12}w_1 - a_{22}w_2 \\ -a_{21}\frac{1}{w_1} & -a_{21}\frac{w_2}{w_1} + a_{22} \end{bmatrix}
$$

Combining these, $B = (dg_{01})g_{10} + g_{01}Ag_{10}$ and

$$
\begin{aligned}
b_{11} &= \frac{2}{w_1}dw_1 + a_{11} + \frac{w_2}{w_1}a_{21} \\
b_{12} &= \frac{w_2}{w_1}dw_1 - dw_2 + w_2(a_{11} - a_{22}) + \frac{w_2^2}{w_1}a_{21} - w_1a_{12} \\
b_{21} &= -\frac{1}{w_1}a_{21} \\
b_{22} &= \frac{1}{w_1}dw_1 - \frac{w_2}{w_1}a_{21} + a_{22}
\end{aligned}
$$

where

$$
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.
$$

Similarly, with

$$
C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}
$$

we obtain

$$
\begin{aligned}
&(dg_{21})g_{12} \\
&= -\tfrac{1}{w_2^3}\begin{bmatrix} dw_2 & 0 \\ -(w_2dw_1 + w_1dw_2) & -2w_2dw_2 \end{bmatrix}\begin{bmatrix} -w_2^2 & 0 \\ w_1w_2 & w_2 \end{bmatrix} \\
&= -\tfrac{1}{w_2^3}\begin{bmatrix} -w_2^2dw_2 & 0 \\ w_2^2(w_2dw_1 + w_1dw_2) - 2w_1w_2^2dw_2 & -2w_2^2dw_2 \end{bmatrix} \\
&= -\tfrac{1}{w_2^3}\begin{bmatrix} -w_2^2dw_2 & 0 \\ w_2^3dw_1 - w_1w_2^2dw_2 & -2w_2^2dw_2 \end{bmatrix}
\end{aligned}
$$

$$
(dg_{21})g_{12} \quad = \quad \begin{bmatrix} \tfrac{1}{w_2}dw_2 & 0 \\ -dw_1 + \tfrac{w_1}{w_2}dw_2 & \tfrac{2}{w_2}dw_2 \end{bmatrix}
$$

$$
\begin{aligned}
&g_{21}Ag_{12} \\
&= -\tfrac{1}{w_2^3}\begin{bmatrix} w_2 & 0 \\ -w_1w_2 & -w_2^2 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} -w_2^2 & 0 \\ w_1w_2 & w_2 \end{bmatrix} \\
&= -\tfrac{1}{w_2^3}\begin{bmatrix} a_{11}w_2 & a_{12}w_2 \\ a_{11}w_1w_2 - a_{21}w_2^2 & -a_{12}w_1w_2 - a_{22}w_2^2 \end{bmatrix}\begin{bmatrix} -w_2^2 & 0 \\ w_1w_2 & w_2 \end{bmatrix} \\
&= -\tfrac{1}{w_2^3}\begin{bmatrix} -a_{11}w_2^3 + a_{12}w_1w_2^2 & a_{21}w_2^2 \\ a_{11}w_1w_2^3 + a_{21}w_2^4 - a_{12}w_1^2w_2^2 - a_{22}w_1w_2^3 & -a_{12}w_1w_2^2 - a_{22}w_2^3 \end{bmatrix}
\end{aligned}
$$

$$
g_{21}Ag_{12} \quad = \quad \begin{bmatrix} a_{11} - a_{12}\tfrac{w_1}{w_2} & -a_{12}\tfrac{1}{w_2} \\ -a_{11}w_1 + a_{21}w_2 - a_{12}\tfrac{w_1^2}{w_2} + a_{22}w_1 & a_{12}\tfrac{w_1}{w_2} + a_{22} \end{bmatrix}
$$

and as $C = (dg_{02})g_{20} + g_{02}Ag_{20}$, the coefficients are

$$
\begin{aligned}
c_{11} &= \frac{1}{w_2}dw_2 + a_{11} - \frac{w_1}{w_2}a_{12} \\
c_{12} &= -\frac{1}{w_2}a_{12} \\
c_{21} &= -dw_1 + \frac{w_1}{w_2}dw_2 + w_1(a_{22} - a_{11}) + \frac{w_1^2}{w_2}a_{12} - w_2a_{21} \\
c_{22} &= \frac{2}{w_2}dw_2 + \frac{w_1}{w_2}a_{12} + a_{22}.
\end{aligned}
$$

4 A connection matrix

The following is an appropriate choice for a connection matrix:

$$
A = -\frac{1}{R_w}\begin{bmatrix} 2\bar{w}_1dw_1 + \bar{w}_2dw_2 & \bar{w}_1dw_2 \\ \bar{w}_2dw_1 & \bar{w}_1dw_1 + 2\bar{w}_2dw_2 \end{bmatrix}
$$

where

$$
R_w = 1 + w_1\bar{w}_1 + w_2\bar{w}_2.
$$

We need to check that this gives rise to a well defined connection matrix on U_0. Recall that

$$\begin{aligned} w_1 &= \frac{1}{z_1} \\ w_2 &= \frac{z_2}{z_1}. \end{aligned}$$

It will be useful to have

$$R_w = 1 + w_1\bar{w}_1 + w_2\bar{w}_2 = 1 + \frac{1}{z_1\bar{z}_1} + \frac{z_2\bar{z}_2}{z_1\bar{z}_1} = \frac{1}{z_1\bar{z}_1}R_z.$$

We also have that

$$\begin{aligned} dw_1 &= \frac{\partial w_1}{\partial z_1}dz_1 + \frac{\partial w_1}{\partial z_2}dz_2 \\ &= -\frac{1}{z_1^2}dz_1 \\ dw_2 &= \frac{\partial w_2}{\partial z_1}dz_1 + \frac{\partial w_2}{\partial z_2}dz_2 \\ &= -\frac{z_2}{z_1^2}dz_1 + \frac{1}{z_1}dz_2. \end{aligned}$$

Now check each coefficient:

$$\begin{aligned} b_{11} &= \frac{2}{w_1}dw_1 + a_{11} + \frac{w_2}{w_1}a_{21} \\ &= \frac{2}{w_1}dw_1 - \frac{1}{R_w}(2\bar{w}_1 dw_1 + \bar{w}_2 dw_2) - \frac{w_2}{w_1}\frac{1}{R_w}\bar{w}_2 dw_1 \\ &= \frac{2}{w_1}dw_1 - \frac{2\bar{w}_1}{R_w}dw_1 - \frac{\bar{w}_2}{R_w}dw_2 - \frac{w_2\bar{w}_2}{w_1 R_w}dw_1 \\ &= -\frac{2}{z_1}dz_1 + \frac{2}{z_1 R_z}dz_1 + \frac{z_2\bar{z}_2}{z_1 R_z}dz_1 - \frac{\bar{z}_2 z_1}{R_z}\left(-\frac{z_2}{z_1^2}dz_1 + \frac{1}{z_1}dz_2\right) \\ &= \frac{-2R_z + 2 + z_2\bar{z}_2 + z_2\bar{z}_2}{z_1 R_z}dz_1 - \frac{\bar{z}_2}{R_z}dz_2 \\ &= -\frac{2\bar{z}_1}{R_z}dz_1 - \frac{\bar{z}_2}{R_z}dz_2 \end{aligned}$$

$$\begin{aligned} b_{12} &= \frac{w_2}{w_1}dw_1 - dw_2 + w_2(a_{11} - a_{22}) + \frac{w_2^2}{w_1}a_{21} - w_1 a_{12} \\ &= \frac{w_2}{w_1}dw_1 - dw_2 + w_2\frac{1}{R_w}\left(-\bar{w}_1 dw_1 + \bar{w}_2 dw_2\right) - \frac{w_2^2}{w_1}\frac{1}{R_w}\bar{w}_2 dw_1 + w_1\frac{1}{R_w}\bar{w}_1 dw_2 \\ &= \frac{w_2}{w_1}dw_1 - dw_2 - \frac{w_2\bar{w}_1}{R_w}dw_1 + \frac{w_2\bar{w}_2}{R_w}dw_2 - \frac{w_2^2\bar{w}_2}{w_1 R_w}dw_1 + \frac{w_1\bar{w}_1}{R_w}dw_2 \\ &= \frac{w_2}{w_1}dw_1 - \frac{w_2\bar{w}_1}{R_w}dw_1 - \frac{w_2^2\bar{w}_2}{w_1 R_w}dw_1 + \left(-1 + \frac{w_2\bar{w}_2}{R_w} + \frac{w_1\bar{w}_1}{R_w}\right)dw_2 \end{aligned}$$

$$\begin{aligned}
&= -\frac{z_2}{z_1^2}dz_1 + \frac{z_2}{z_1^2 R_z}dz_1 + \frac{z_2^2\bar{z}_2}{z_1^2 R_z}dz_1 + \left(-1 + \frac{1}{R_z} + \frac{z_2\bar{z}_2}{R_z}\right)\left(-\frac{z_2}{z_1^2}dz_1 + \frac{1}{z_1}dz_2\right) \\
&= \frac{-R_z + 1 + z_2\bar{z}_2}{z_1 R_z}dz_2 \\
&= -\frac{\bar{z}_1}{R_z}dz_2
\end{aligned}$$

$$\begin{aligned}
b_{21} &= -\frac{1}{w_1}a_{21} \\
&= \frac{1}{w_1}\frac{\bar{w}_2}{R_w}dw_1 \\
&= -\frac{\bar{z}_2}{R_z}dz_1
\end{aligned}$$

$$\begin{aligned}
b_{22} &= \frac{1}{w_1}dw_1 - \frac{w_2}{w_1}a_{21} + a_{22} \\
&= \frac{1}{w_1}dw_1 + \frac{w_2}{w_1}\frac{1}{R_w}\bar{w}_2 dw_1 - \frac{1}{R_w}(\bar{w}_1 dw_1 + 2\bar{w}_2 dw_2) \\
&= \frac{1}{w_1}dw_1 + \frac{w_2\bar{w}_2}{w_1 R_w}dw_1 - \frac{\bar{w}_1}{R_w}dw_1 - \frac{2\bar{w}_2}{R_w}dw_2 \\
&= -\frac{1}{z_1}dz_1 - \frac{z_2\bar{z}_2}{z_1 R_z}dz_1 + \frac{1}{z_1 R_z}dz_1 - \frac{\bar{z}_2 z_1}{R_z}\left(-\frac{z_2}{z_1^2}dz_1 + \frac{1}{z_1}dz_2\right) \\
&= \frac{-R_z - z_2\bar{z}_2 + 2z_2\bar{z}_2 + 1}{z_1 R_z}dz_1 - \frac{2\bar{z}_2}{R_z}dz_2 \\
&= -\frac{\bar{z}_1}{R_z}dz_1 - \frac{2\bar{z}_2}{R_z}dz_2
\end{aligned}$$

Now

$$B = -\frac{1}{R_z}\begin{bmatrix} 2\bar{z}_1 dz_1 + \bar{z}_2 dz_2 & \bar{z}_1 dz_2 \\ \bar{z}_2 dz_1 & \bar{z}_1 dz_1 + 2\bar{z}_2 dz_2 \end{bmatrix}$$

where

$$R_z = 1 + z_1\bar{z}_1 + z_2\bar{z}_2.$$

This shows that A and B agree on the overlap U_{01}, and that this B is defined for all $(x_1, x_2) \in \mathbb{C}^2$, hence on all of U_0, as required. To see that the same is true for C, let $\tilde{a}_{ij}$ and $\tilde{c}_{ij}$ denote the coefficients of A and C with the w_1 and w_2 switched. Then notice that

$$B = \begin{bmatrix} \tilde{c}_{22} & \tilde{c}_{21} \\ \tilde{c}_{12} & \tilde{c}_{11} \end{bmatrix}$$

while

$$A = \begin{bmatrix} \tilde{a}_{22} & \tilde{a}_{21} \\ \tilde{a}_{12} & \tilde{a}_{11} \end{bmatrix}.$$

Now since

$$w_1 = \frac{1}{z_1} = \frac{y_1}{y_2}$$
$$w_2 = \frac{z_2}{z_1} = \frac{1}{y_2}$$

a computation, similar to the one above (with the roles of w_1 and w_2 exchanged), shows that

$$C = -\frac{1}{R_y}\begin{bmatrix} 2\bar{y}_1 dy_1 + \bar{y}_2 dy_2 & \bar{y}_1 dy_2 \\ \bar{y}_2 dy_1 & \bar{y}_1 dy_1 + 2\bar{y}_2 dy_2 \end{bmatrix}$$

where

$$R_y = 1 + y_1\bar{y}_1 + y_2\bar{y}_2.$$

5 The curvature F_A

We need to compute

$$F_A = dA + A \wedge A.$$

Set $R_w = R$ so that

$$dR = w_1 d\bar{w}_1 + \bar{w}_1 dw_1 + w_2 d\bar{w}_2 + \bar{w}_2 dw_2$$

and

$$\begin{aligned} dA &= \frac{dR}{R^2}\begin{bmatrix} 2\bar{w}_1 dw_1 + \bar{w}_2 dw_2 & \bar{w}_1 dw_2 \\ \bar{w}_2 dw_1 & \bar{w}_1 dw_1 + 2\bar{w}_2 dw_2 \end{bmatrix} - \frac{1}{R}\begin{bmatrix} 2d\bar{w}_1 dw_1 + d\bar{w}_2 dw_2 & d\bar{w}_1 dw_2 \\ d\bar{w}_2 dw_1 & d\bar{w}_1 dw_1 + 2d\bar{w}_2 dw_2 \end{bmatrix} \\
&= \frac{1}{R^2}\begin{bmatrix} \begin{array}{l} 2w_1\bar{w}_1 d\bar{w}_1 dw_1 + w_1\bar{w}_2 d\bar{w}_1 dw_2 + \bar{w}_1\bar{w}_2 dw_1 dw_2 \\ \quad +2\bar{w}_1 w_2 d\bar{w}_2 dw_1 + w_2\bar{w}_2 d\bar{w}_2 dw_2 + 2\bar{w}_1\bar{w}_2 dw_2 dw_1 \end{array} & w_1\bar{w}_1 d\bar{w}_1 dw_2 + \bar{w}_1^2 dw_1 dw_2 + w_2\bar{w}_1 d\bar{w}_2 dw_2 \\ w_1\bar{w}_2 d\bar{w}_1 dw_1 + w_2\bar{w}_2 d\bar{w}_2 dw_1 + \bar{w}_2^2 dw_2 dw_1 & \begin{array}{l} w_1\bar{w}_1 d\bar{w}_1 dw_1 + 2w_1\bar{w}_2 d\bar{w}_1 dw_2 + 2\bar{w}_1\bar{w}_2 dw_1 dw_2 \\ \quad +w_2\bar{w}_1 d\bar{w}_2 dw_1 + 2w_2\bar{w}_2 d\bar{w}_2 dw_2 + \bar{w}_1\bar{w}_2 dw_2 dw_1 \end{array} \end{bmatrix} \\
&\quad + \frac{R}{R^2}\begin{bmatrix} 2dw_1 d\bar{w}_1 + dw_2 d\bar{w}_2 & -d\bar{w}_1 dw_2 \\ dw_1 d\bar{w}_2 & dw_1 d\bar{w}_1 + 2dw_2 d\bar{w}_2 \end{bmatrix} \\
&= \frac{1}{R^2}\begin{bmatrix} -2w_1\bar{w}_1 dw_1 d\bar{w}_1 + w_1\bar{w}_2 d\bar{w}_1 dw_2 - \bar{w}_1\bar{w}_2 dw_1 dw_2 - 2\bar{w}_1 w_2 dw_1 d\bar{w}_2 - w_2\bar{w}_2 dw_2 d\bar{w}_2 & w_1\bar{w}_1 d\bar{w}_1 dw_2 + \bar{w}_1^2 dw_1 dw_2 - \bar{w}_1 w_2 dw_2 d\bar{w}_2 \\ -w_1\bar{w}_2 dw_1 d\bar{w}_1 - w_2\bar{w}_2 dw_1 d\bar{w}_2 - \bar{w}_2^2 dw_1 dw_2 & -w_1\bar{w}_1 dw_1 d\bar{w}_1 + 2w_1\bar{w}_2 d\bar{w}_1 dw_2 + \bar{w}_1\bar{w}_2 dw_1 dw_2 - \bar{w}_1 w_2 dw_1 d\bar{w}_2 - 2w_2\bar{w}_2 dw_2 d\bar{w}_2 \end{bmatrix} \\
&\quad + \frac{1}{R^2}\begin{bmatrix} R(2dw_1 d\bar{w}_1 + dw_2 d\bar{w}_2) & -Rd\bar{w}_1 dw_2 \\ Rdw_1 d\bar{w}_2 & R(dw_1 d\bar{w}_1 + 2dw_2 d\bar{w}_2) \end{bmatrix} \end{aligned}$$

$$= \frac{1}{R^2}\begin{bmatrix} \begin{matrix}(2+2w_2\bar{w}_2)dw_1 d\bar{w}_1 + w_1\bar{w}_2 d\bar{w}_1 dw_2 - \bar{w}_1\bar{w}_2 dw_1 dw_2 \\ -2\bar{w}_1 w_2 dw_1 d\bar{w}_2 + (1+w_1\bar{w}_1)dw_2 d\bar{w}_2\end{matrix} & (-1-w_2\bar{w}_2)d\bar{w}_1 dw_2 + \bar{w}^2 dw_1 dw_2 - \bar{w}_1 w_2 dw_2 d\bar{w}_2 \\ -w_1\bar{w}_2 dw_1 d\bar{w}_1 + (1+w_1\bar{w}_1)dw_1 d\bar{w}_2 - \bar{w}_2^2 dw_1 dw_2 & \begin{matrix}(1+w_2\bar{w}_2)dw_1 d\bar{w}_1 + 2w_1\bar{w}_2 d\bar{w}_1 dw_2 + \bar{w}_1\bar{w}_2 dw_1 dw_2 \\ -\bar{w}_1 w_2 dw_1 d\bar{w}_2 + (2+2w_1\bar{w}_1)dw_2 d\bar{w}_2\end{matrix}\end{bmatrix}$$

$$\begin{aligned}
A\wedge A &= \frac{1}{R^2}\left(\begin{bmatrix} 2\bar{w}_1 dw_1 + \bar{w}_2 dw_2 & \bar{w}_1 dw_2 \\ \bar{w}_2 dw_1 & \bar{w}_1 dw_1 + 2\bar{w}_2 dw_2 \end{bmatrix} \wedge \begin{bmatrix} 2\bar{w}_1 dw_1 + \bar{w}_2 dw_2 & \bar{w}_1 dw_2 \\ \bar{w}_2 dw_1 & \bar{w}_1 dw_1 + 2\bar{w}_2 dw_2 \end{bmatrix}\right) \\
&= \frac{1}{R^2}\begin{bmatrix} (2\bar{w}_1 dw_1 + \bar{w}_2 dw_2)^2 + (\bar{w}_1 dw_2)(\bar{w}_2 dw_1) & (2\bar{w}_1 dw_1 + \bar{w}_2 dw_2)\bar{w}_1 dw_2 + \bar{w}_1 dw_2(\bar{w}_1 dw_1 + 2\bar{w}_2 dw_2) \\ \bar{w}_2 dw_1(2\bar{w}_1 dw_1 + \bar{w}_2 dw_2) + (\bar{w}_1 dw_1 + 2\bar{w}_2 dw_2)\bar{w}_2 dw_1 & (\bar{w}_2 dw_1)(\bar{w}_1 dw_2) + (\bar{w}_1 dw_1 + 2\bar{w}_2 dw_2)^2 \end{bmatrix} \\
&= \frac{1}{R^2}\begin{bmatrix} \bar{w}_1\bar{w}_2 dw_2 dw_1 & 2\bar{w}_1^2 dw_1 dw_2 + \bar{w}_1^2 dw_2 dw_1 \\ \bar{w}_2^2 dw_1 dw_2 + 2\bar{w}_2^2 dw_2 dw_1 & \bar{w}_1\bar{w}_2 dw_1 dw_2 \end{bmatrix} \\
&= \frac{1}{R^2}\begin{bmatrix} -\bar{w}_1\bar{w}_2 dw_1 dw_2 & \bar{w}_1^2 dw_1 dw_2 \\ -\bar{w}_2^2 dw_1 dw_2 & \bar{w}_1\bar{w}_2 dw_1 dw_2 \end{bmatrix}.
\end{aligned}$$

Now

$$F_A = \frac{1}{R^2}\begin{bmatrix} \begin{matrix}(2+2w_2\bar{w}_2)dw_1 d\bar{w}_1 + w_1\bar{w}_2 d\bar{w}_1 dw_2 - 2\bar{w}_1\bar{w}_2 dw_1 dw_2 \\ -2\bar{w}_1 w_2 dw_1 d\bar{w}_2 + (1+w_1\bar{w}_1)dw_2 d\bar{w}_2\end{matrix} & (-1-w_2\bar{w}_2)d\bar{w}_1 dw_2 + 2\bar{w}^2 dw_1 dw_2 - \bar{w}_1 w_2 dw_2 d\bar{w}_2 \\ -w_1\bar{w}_2 dw_1 d\bar{w}_1 + (1+w_1\bar{w}_1)dw_1 d\bar{w}_2 - 2\bar{w}_2^2 dw_1 dw_2 & \begin{matrix}(1+w_2\bar{w}_2)dw_1 d\bar{w}_1 + 2w_1\bar{w}_2 d\bar{w}_1 dw_2 + 2\bar{w}_1\bar{w}_2 dw_1 dw_2 \\ -\bar{w}_1 w_2 dw_1 d\bar{w}_2 + (2+2w_1\bar{w}_1)dw_2 d\bar{w}_2\end{matrix}\end{bmatrix}$$

6 Chern Numbers

Finally, we can compute

$$\int_{\mathbb{CP}^2}\left(Tr\left[\frac{iF_A}{2\pi}\right]\right)^2$$

and

$$\int_{\mathbb{CP}^2} Tr\left[\frac{iF_A}{2\pi}\right]^2.$$

First,

$$Tr\left[\frac{iF_A}{2\pi}\right] = \frac{i}{2\pi R^2}\Big((3+3w_2\bar{w}_2)dw_1d\bar{w}_1 + 3w_1\bar{w}_2d\bar{w}_1dw_2 - 3\bar{w}_1w_2dw_1d\bar{w}_2 + (3+3w_1\bar{w}_1)dw_2d\bar{w}_2\Big)$$

so that

$$\begin{aligned}
\left(Tr\left[\frac{iF_A}{2\pi}\right]\right)^2 &= -\frac{1}{4\pi^2R^4}\left(\begin{array}{l}(3+3w_2\bar{w}_2)(3+3w_1\bar{w}_1)dw_1d\bar{w}_1dw_2d\bar{w}_2\\ \quad -9w_1\bar{w}_1w_2\bar{w}_2d\bar{w}_1dw_2dw_1d\bar{w}_2\\ \quad\quad -9w_1\bar{w}_1w_2\bar{w}_2dw_1d\bar{w}_2d\bar{w}_1dw_2\\ \quad\quad\quad +(3+3w_1\bar{w}_1)(3+3w_2\bar{w}_2)dw_2d\bar{w}_2dw_1d\bar{w}_1\end{array}\right)\\
&= -\frac{1}{4\pi^2R^4}\Big(2(3+3w_1\bar{w}_1)(3+3w_2\bar{w}_2) - 18w_1\bar{w}_1w_2\bar{w}_2\Big)dw_1d\bar{w}_1dw_2d\bar{w}_2\\
&= -\frac{18}{4\pi^2R^4}\Big(1+w_1\bar{w}_1+w_2\bar{w}_2\Big)dw_1d\bar{w}_1dw_2d\bar{w}_2\\
&= -\frac{9}{2\pi^2R^3}dw_1d\bar{w}_1dw_2d\bar{w}_2.
\end{aligned}$$

Second,

$$Tr\left[\frac{iF_A}{2\pi}\right]^2 = -\frac{1}{4\pi^2R^4}\left(\begin{array}{l}\Big((2+2w_2\bar{w}_2)dw_1d\bar{w}_1 + w_1\bar{w}_2d\bar{w}_1dw_2 - 2\bar{w}_1\bar{w}_2dw_1dw_2\\ \quad -2\bar{w}_1w_2dw_1d\bar{w}_2 + (1+w_1\bar{w}_1)dw_2d\bar{w}_2\Big)^2\\ \\ +\Big((-1-w_2\bar{w}_2)d\bar{w}_1dw_2 + 2\bar{w}^2dw_1dw_2 - \bar{w}_1w_2dw_2d\bar{w}_2\Big)\\ \quad\Big(-w_1\bar{w}_2dw_1d\bar{w}_1 + (1+w_1\bar{w}_1)dw_1d\bar{w}_2 - 2\bar{w}_2^2dw_1dw_2\Big)\\ \\ +\Big(-w_1\bar{w}_2dw_1d\bar{w}_1 + (1+w_1\bar{w}_1)dw_1d\bar{w}_2 - 2\bar{w}_2^2dw_1dw_2\Big)\\ \quad\Big((-1-w_2\bar{w}_2)d\bar{w}_1dw_2 + 2\bar{w}^2dw_1dw_2 - \bar{w}_1w_2dw_2d\bar{w}_2\Big)\\ \\ +\Big((1+w_2\bar{w}_2)dw_1d\bar{w}_1 + 2w_1\bar{w}_2d\bar{w}_1dw_2 + 2\bar{w}_1\bar{w}_2dw_1dw_2\\ \quad -\bar{w}_1w_2dw_1d\bar{w}_2 + (2+2w_1\bar{w}_1)dw_2d\bar{w}_2\Big)^2\end{array}\right)$$

$$\begin{aligned}
&= -\frac{1}{4\pi^2R^4}\Big(6(1+w_2\bar{w}_2)(1+w_1\bar{w}_1)dw_1d\bar{w}_1dw_2d\bar{w}_2 - 6w_1\bar{w}_1w_2\bar{w}_2dw_1d\bar{w}_1dw_2d\bar{w}_2\Big)\\
&= -\frac{6}{4\pi^2R^4}\Big(1+w_1\bar{w}_1+w_2\bar{w}_2\Big)dw_1d\bar{w}_1dw_2d\bar{w}_2\\
&= -\frac{3}{2\pi^2R^3}dw_1d\bar{w}_1dw_2d\bar{w}_2.
\end{aligned}$$

The following integration will be useful for both cases:

$$\begin{aligned}
\frac{2}{\pi^2}\int_{\mathbb{R}^4}\frac{dx_1dx_2dy_1dy_2}{(1+x_1^2+y_1^2+x_2^2+y_2^2)^3} &= \frac{2}{\pi^2}\int_0^\infty\frac{r^3dr}{(1+r^2)^3}vol\mathbb{S}^3\\
&= 4\int_0^\infty\frac{r^3}{(1+r^2)^3}dr
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_1^{\infty} \frac{u-1}{u^3} du \qquad\qquad \text{where} \quad u = 1+r^2 \\
&= 2\int_1^{\infty} \left(\frac{1}{u^2} - \frac{1}{u^3}\right) du \\
&= 2\left[-\frac{1}{u} + \frac{1}{2u^2}\right]_1^{\infty} \\
&= 1.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_{\mathbb{CP}^2} \left(Tr\left[\frac{iF_A}{2\pi}\right]\right)^2 &= -\int_{U_1} \frac{9}{2\pi^2 R^3} dw_1 d\bar{w}_1 dw_2 d\bar{w}_2 \\
&= -\frac{9}{2\pi^2}\int_{\mathbb{C}^2} \frac{dw_1 d\bar{w}_1 dw_2 d\bar{w}_2}{(1+w_1\bar{w}_1 + w_2\bar{w}_2)^3} \\
&= -\frac{9}{2\pi^2}\int_{\mathbb{R}^4} \frac{-4dx_1 dy_1 dx_2 dy_2}{(1+x_1^2+y_1^2+x_2^2+y_2^2)^3} \\
&= 9\left(\frac{2}{\pi^2}\int_{\mathbb{R}^4} \frac{dx_1 dy_1 dx_2 dy_2}{(1+x_1^2+y_1^2+x_2^2+y_2^2)^3}\right) \\
&= 9
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\mathbb{CP}^2} Tr\left[\frac{iF_A}{2\pi}\right]^2 &= -\int_{U_1} \frac{3}{2\pi^2 R^3} dw_1 d\bar{w}_1 dw_2 d\bar{w}_2 \\
&= -\frac{3}{2\pi^2}\int_{\mathbb{C}^2} \frac{dw_1 d\bar{w}_1 dw_2 d\bar{w}_2}{(1+w_1\bar{w}_1 + w_2\bar{w}_2)^3} \\
&= -\frac{3}{2\pi^2}\int_{\mathbb{R}^4} \frac{-4dx_1 dy_1 dx_2 dy_2}{(1+x_1^2+y_1^2+x_2^2+y_2^2)^3} \\
&= 3\left(\frac{2}{\pi^2}\int_{\mathbb{R}^4} \frac{dx_1 dy_1 dx_2 dy_2}{(1+x_1^2+y_1^2+x_2^2+y_2^2)^3}\right) \\
&= 3.
\end{aligned}
$$