COMPUTATIONS IN HEEGAARD-FLOER HOMOLOGY

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The goal of this talk is to illustrate some of the subtleties in the definition of the Heegaard-Floer complex $\widehat{CF}(Y)$ defined in [7]. In particular, we will look at the simple case of the sphere where $\widehat{HF}(S^3) = \mathbb{Z}_2$. The survey articles [5, 8, 9, 10] are a good place to start.

1. Background

Let Y be a smooth, compact, connected, orientable 3-manifold.

Heegaard Diagrams. A handlebody is a regular neighborhood of a connected graph. For example, the regular neighbourhood of a tree gives rise to a 3-ball. In general, the result is orientable 3-manifold with boundary a genus g surface. Thought of in this way, a handlebody can be obtained from a 3-ball by attaching g handles $D^2 \times [0,1]$ to the ball (via a homeomorphism $h : D^2 \times \{0,1\} \to \partial B^3$) by specifying $2g$ disjoint disks in the boundary of the ball.

Given two handle bodies of the same genus, a 3-manifold my be specified by identifying the boundaries via homeomorphism; a Heegaard decomposition of Y is a splitting of Y along a genus g surface.

Consider a self-indexing Morse function $f : Y \to \mathbb{R}$ with exactly one index 0 critical point as well as exactly one index 3 critical point. Since $\chi(Y) = 0$ there must be the same number of index 1 as index 2 critical points. In particular, $f^{-1}(\frac{3}{2})$ is an orientable surface of genus $g > 0$ (in the case of $S^3 = e^0 \cup e^3$, we can add a canceling pair of handles $e^0 \cup e^1 \cup e^2 \cup e^3$ to ensure $g \neq 0$). Moreover, f decomposes Y into handlebodies $Y = U_0 \cup U_1$ where $U_0 = f^{-1}[0, \frac{3}{2}]$ and $U_1 = f^{-1}[\frac{3}{2}, 1]$.

A Heegaard diagram for Y is a triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ where

$$\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$$

is a set of attaching curves on Σ for the 1-handles specified by the index 1 critical points, and

$$\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$$

is a set of attaching curves on Σ for the 2-handles specified by the index 2 critical points. As a result

$$H_1(Y; Z) = \frac{H_1(\Sigma; \mathbb{Z})}{\langle [\alpha_1], \dots, [\alpha_g], [\beta_1], \dots [\beta_g] \rangle}.$$

By varying the Morse function used in the construction it follows that Y admits many different Heegaard diagrams. The following theorem [14] tells us how to move between these various descriptions, and in particular what needs to be checked for any invariant of Y defined via Heegaard diagrams.

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Theorem (Reidemeister, Singer). *Two Heegaard diagrams represent the same 3-manifold if the are related by a sequence of (1) istopies, (2) stabilizations, and (3) handle-slides (see section 3 for examples of these moves).*

The definition of Heegaard-Floer homology makes use of marked Heegaard diagrams $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, where $z \in \Sigma \smallsetminus \boldsymbol{\alpha} \smallsetminus \boldsymbol{\beta}$, for which a similar statement holds [7].

Symmetric Products. The g-fold symmetric product of a genus g surface Σ is given by

$$\mathrm{Sym}^g \Sigma = \overbrace{\Sigma \times \cdots \times \Sigma}^{g} \Big/ S_g$$

where S_g is the symmetric group on g letters acting by permuting the coordinates. Although the action is clearly not free, it can be shown that $\mathrm{Sym}^g \Sigma$ is a complex manifold. For example, in genus 1 $\mathrm{Sym}^1 \Sigma = \Sigma$ (obvious) and in genus 2 $\mathrm{Sym}^2 \Sigma = T^4 \# \overline{\mathbb{C}P^2}$ (not obvious, see [1]). Moreover, Perutz [11] has shown that $\mathrm{Sym}^g \Sigma$ is a symplectic manifold, and that the natural tori

$$\mathbf{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$$

and

$$\mathbf{T}_\beta = \beta_1 \times \cdots \times \beta_g$$

are Lagrangian. By isotopy of the surface Σ, we may assume that the intersection $\mathbf{T}_\alpha \cap \mathbf{T}_\beta$ is transverse. It can be shown that

$$H_1(Y; Z) = \frac{H_1(\Sigma; \mathbb{Z})}{\langle [\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots [\beta_g] \rangle} = \frac{H_1(\mathrm{Sym}^g \Sigma; \mathbb{Z})}{H_1(\mathbf{T}_\alpha; \mathbb{Z}) \oplus H_1(\mathbf{T}_\beta; \mathbb{Z})}.$$

Symmetric products are studied extensively in [4].

Holomorphic Discs. Let D^2 be the standard unit disk in $\mathbb{C}$. For intersection points $\mathbf{x}, \mathbf{y} \in \mathbf{T}_\alpha \cap \mathbf{T}_\beta$ let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the homotopy classes of whitney discs from $\mathbf{x}$ to $\mathbf{y}$. That is

$$\pi_2(\mathbf{x}, \mathbf{y}) = \left\{ \phi : \mathbb{D}^2 \to \mathrm{Sym}^g \Sigma \;\middle|\; \begin{array}{c} \phi(-i) = \mathbf{x} \\ \phi(i) = \mathbf{y} \\ \phi(e^+) \subset \mathbf{T}_\alpha \\ \phi(e^-) \subset \mathbf{T}_\beta \end{array} \right\}$$

where e^+ is the positive real part of $\partial \mathbb{D}^2$ and e^- is the negative real part of $\partial \mathbb{D}^2$.

When ϕ admits a holomorphic representative, we can denote the Maslov index of ϕ by $\mu(\phi)$. This is the expected dimension of the moduli space $\mathcal{M}(\phi)$. There is a natural $\mathbb{R}$ action on D^2 fixing $\pm i$ so that according to Gromov [2], $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi) \big/ \mathbb{R}$ is a finite number of points whenever $\mu(\phi) = 1$.

2. Definition

Fix a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for Y. For the purpose of this talk we make the further simplifying assumption that $H_1(Y; \mathbb{Z}) = 0$ and take coefficients in Z_2 for the Heegaard-Floer complex. Let $\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be the free Z_2 module generated by intersection points $\mathbf{T}_\alpha \cap \mathbf{T}_\beta$ with

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbf{T}_\alpha \cap \mathbf{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1 \\ n_z(\phi) = 0}} \#_2 \widehat{\mathcal{M}}(\phi) \mathbf{y}.$$

Here, $n_z(\phi)$ is the algebraic intersection with the complex codimension 1 submanifold

$$\{z\} \times \mathrm{Sym}^{g-1}\Sigma \subset \mathrm{Sym}^g \Sigma.$$

The definition of ∂ depends on a choice of complex structure for Σ, and a path of nearly symmetric almost complex structures on $\mathrm{Sym}^g \Sigma$.

The following are proved proved in [7]:

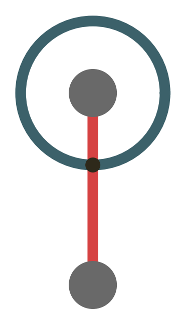
Theorem (Ozsváth-Szabó, Gromov, Floer, ...). *There exist generic choices so that* $\partial \circ \partial = 0$.

Theorem (Ozsváth-Szabó). *The homology* $\widehat{HF}(Y) = H_*(\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z), \partial)$ *is an invariant of the maniold* Y *specified by the pointed Heegaard diagram* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$.

3. Examples

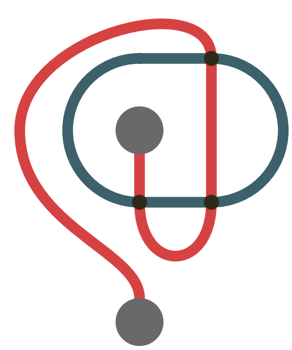
The main goal is to investigate some of the subtleties of Ozsváth and Szabó's theorem by looking at a sequence of Heegaard diagrams for S^3. This section is based on an example from [8]; the techniques used throughout originate (to the best of my knowledge) in [3, 7, 6, 12].

The convention used below should is as follows. The page constitutes the Heegaard surface Σ, together with handles attached (vertically, in the cases where there is ambiguity) to the gray discs. The red curves specify the $\boldsymbol{\alpha}$ (adding no new twists when passing ove the handle) while the blue curves specify the $\boldsymbol{\beta}$. In this way, the index 0 and 1 critical points lie above the page, while the index 2 and 3 critical points lie below. Finally, the marked point z is taken somewhere away from the curves specified (and does not appear in any of the pictures below).



To begin, we start with the simplest (genus 1) Heegaard diagram for S^3. This can be thought of as the handle decomposition $e^0 \cup e^1 \cup e^2 \cup e^3$ for S^3 where the pair (e^1, e^2) cancel. It should be clear that α_1 bounds a disc in U_0 (above the page) and that β_1 bounds a disk in U_1 (below the page), More generally, a stabilization raises (or lowers) the genus of Σ by adding (or removing) such a canceling pair.

Recall that $\mathrm{Sym}^1 \Sigma = \Sigma$, and from the picture we see that there is a unique intersection point $\mathbf{T}_\alpha \cap \mathbf{T}_\beta = \alpha_1 \cap \beta_1$ generating $\widehat{CF}(S^3) = \mathbb{Z}_2$. There are no differentials to count, and we obtain $\widehat{HF}(S^3) = \mathbb{Z}_2$.

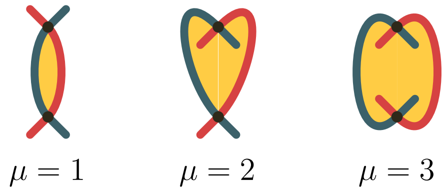


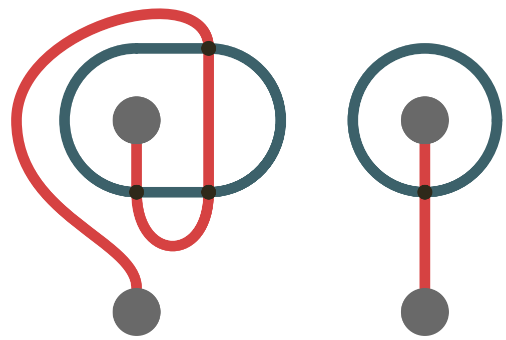
By isotopy, we may further complicate this picture and add generator (which necessarily cancel in homology. One such example with 3 generators is given on the right, where $\widehat{CF}(S^3) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

The curves given divide the torus into 3 regions: One containing the marked point z (which we discard) and 2 bigons (which must provide non-trivial differentials). Since locally we are working in $\mathbb{C}$in this setting, we may apply the Riemann mapping theorem to conclude that there is a unique holomorphic representative (up to $\mathbb{R}$-action) for each bigon, hence $\mu = 1$ in both cases. In particular, for an appropriate labeling, $\partial x_1 = x_2$ and $\partial x_3 = x_2$ and the complex has the form

$$\begin{array}{ccc} x_1 & & x_3 \\ & \searrow \quad \swarrow & \\ & x_2 & \end{array}$$

and $\widehat{HF}(S^3) = Z_2$ (genrated by $x_1 + x_3$) as required. Note however that not all bigons give this result, and that in fact (by adding cuts) we have



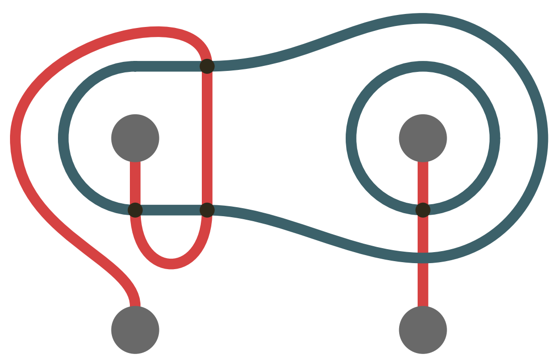


Now consider a stabilization of the previous example to obtain a a genus 2 Heegaard diagram for S^3. We now have (x_1, y), (x_2, y) and (x_3, y) generating $\widehat{CF}(S^3) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Again we need to argue that the the bigons appearing in the surface correspond to discs in $\mathrm{Sym}^2 \Sigma$ and in particular generate the same maps as before.

Let $\mathcal{D}$ be the domain in Σ consisting of pairs (b, y) where b is a point in the disc spanned by the bigon, and y is the intersection point added by the stabilization. Note that S_2 acts freely on $\mathcal{D}$, and we can think of the class of points $(b, y) \sim (y, b)$ as points in $\mathrm{Sym}^g \Sigma$. It follows that each of the two domains $\mathcal{D}$ contribute differentials as before so that the complex

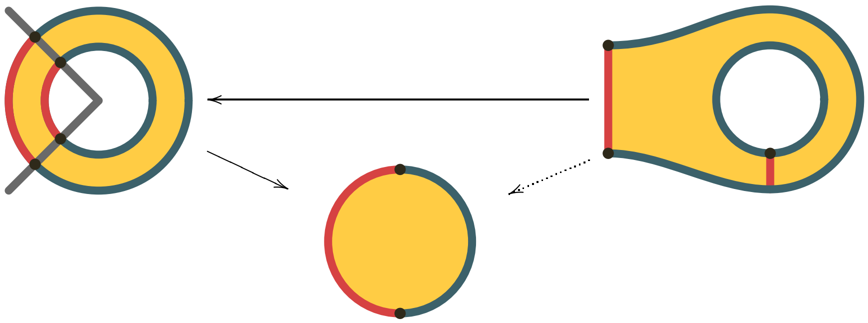
$$\begin{array}{ccc} (x_1, y) & & (x_3, y) \\ & \searrow \quad \swarrow & \\ & (x_2, y) & \end{array}$$

gives $\widehat{HF}(S^3) = \mathbb{Z}_2$ as before.

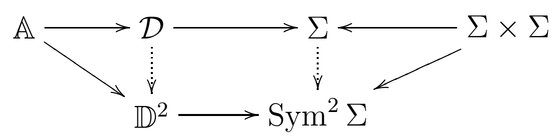


A handle-slide between two curves in $\boldsymbol{\beta}$ (equivalently curves in $\boldsymbol{\alpha}$) exists whenever there is a third curve in $\boldsymbol{\beta}$ such that the triple bounds a pair of pants in Σ. An example of a handle-slide aplied to the last example is given here (in this case there is an obvious pair of pants in the plane bounded by the three curves in question).

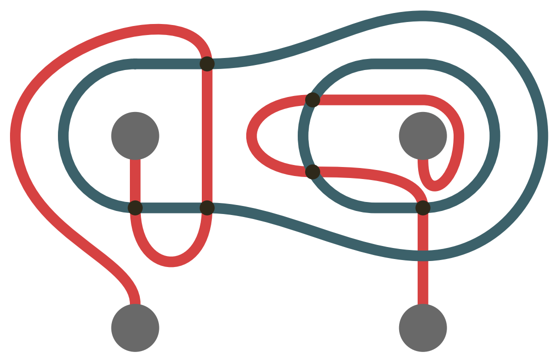
In this case we have that $\partial(x_1, y) = (x_2, y)$ as before, however the second bigon has now been replaced by an annular domain $\mathcal{D}$. Let $\mathbb{A}$ be the standard annulus in $\mathbb{C}$ and recall that $\mathbb{A}$ is uniformized by the ratio of the outer and inner radii. Therefore there is precisely one length of cut for which the image $\mathcal{D} \to \mathbb{A}$ admits an involution $z \to \frac{1}{z}$ which fixes the generators (x_2, y) and (x_3, y) (it will however exchange the coordinates).



That is, adding a cut along α_2 begining at y we have exactly 1 annuli that branch covers the disc. The key observation is that the domain $\mathcal{D} \hookrightarrow \Sigma$ together with the 2-fold branched cover $\mathcal{D} \searrow \mathbb{D}^2$ gives rise to a holomorphic disc in $\mathrm{Sym}^2\,\Sigma$.

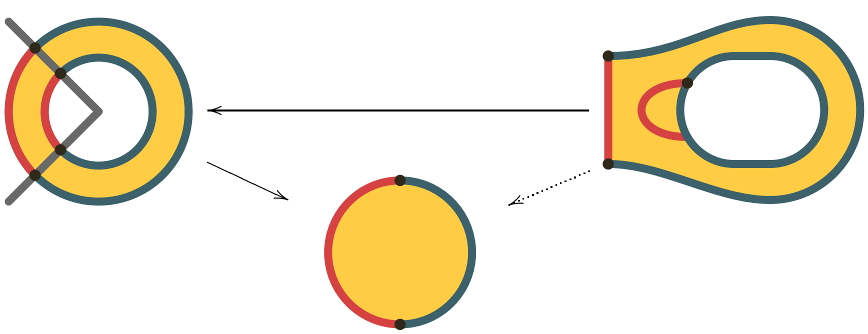


We can further complicate the situation by twisting. In this case we still get S^3 only the stabilization (before the handle-slide) uses a more complicated description of S^3. This particular example is what the entire talk is based on; it is found in [8]. In particular, it illustrates the potential dependence of the complex $\widehat{CF}(Y)$ on the complex structure on Σ.

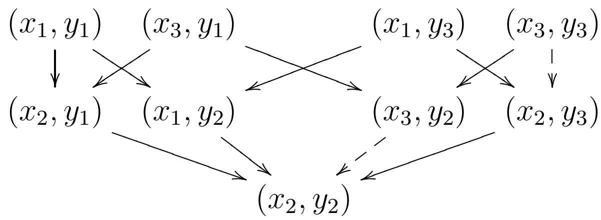


We continue with the same labels x_1, x_2, x_3 and label the new points clockwise y_1, y_2, y_3 where y_1 was formerly y. There the complex now has 9 generators, and as before, we have bigons providing differentials $\partial(x_1, y_i) = (x_2, y_i)$, $\partial(x_3, y_i) = (x_2, y_i)$, $\partial(x_i, y_1) = (x_i, y_2)$ and $\partial(x_i, y_3) = (x_i, y_1)$, as well as the differential $\partial(x_3, y_1) = (x_2, y_1)$ provided by the annular domain of the previous example. It is easy to check however, that more differentials are required to obtain the correct homology.

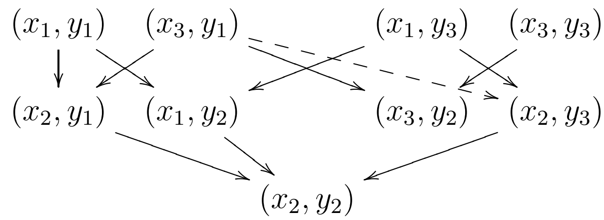
First, it should be checked that there is no differential $\partial(x_3, y_2) = (x_2, y_3)$ provided by the corresponding annular domain (look at the corners of the y_i). This leaves two annuli giving $\partial(x_3, y_{2,3}) = (x_2, y_{2,3})$ and another giving $\partial(x_3, y_3) = (x_2, y_1)$. The fact is that we get the first situation or the second depending on the complex structure on Σ; the domain for $\partial(x_3, y_3) = (x_2, y_3)$ is shown below.



Let a be the angle spanned by the arc (of the image of α_1 in $\mathbb{A}$) between x_3 and x_2, and let b be the length of the corresponding arc (of the image of α_2) between y_3 and y_2. Then the annulus above gives $\partial(x_3, y_3) = (x_2, y_3)$ whenever $a < b$. Indeed, we have the complexes

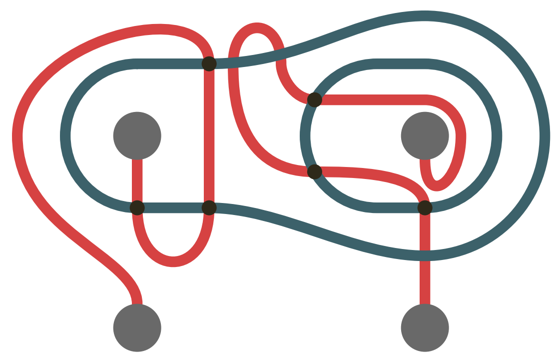


$a < b$

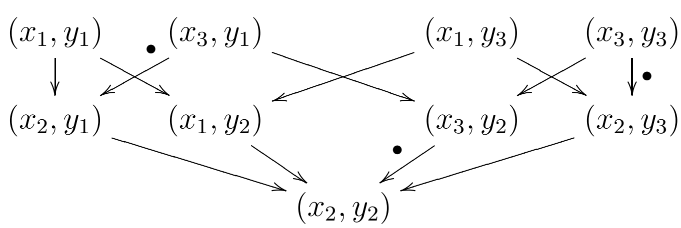


$a > b$

which each give the desired homology.



The examples given thus far show (or at least, begin to suggest) that there are techniques available to treat explicit computations. The surprising fact is that a combinatorial description for this theory now exists [13]. This final example shows that we can eliminate the dependence on the complex structure for this particular case by an isotopy. The isotopy illustrated does not add any new generators, but it simplifies all the annular regions corresponding to differentials, eliminating the behavior from the previous example. The corresponding chain complex is given below; the arrows marked by $\bullet$ correspond to differentials that are generated by annular domains.



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