Generalizing Poincaré-Type UQAM Kähler Metrics

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Setting: (X^n, σ, J, g) compact Kähler mfld

$\dim_{\mathbb{C}} X = n$

$$\begin{tikzcd}
TX \arrow[rr, "J"] & & TX \arrow[dl, "g"] \\
& T^*X \arrow[ul, "\sigma"] &
\end{tikzcd}$$

$J^2 = -\mathrm{id}$

σ symplectic

g Riemannian

$U(n)$-Structure integrable to 1^{st} order $\iff \nabla J = 0$

Consequences: $T_{\mathbb{C}} X = T^{1,0}X \oplus T^{0,1}X$

$$\Omega^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X)$$

$d = \partial + \bar{\partial}$ $\qquad$ $\bar{\partial}: \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$

$\bar{\partial}: \Omega^{q}(T^{1,0}) \to \Omega^{q+1}(T^{1,0})$

$d^c = J d J^{-1}$ $\qquad$ $\bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$

$dd^c = 2i\bar{\partial}\partial$ $\qquad$ $\mathcal{D} = \bar{\partial}(\bar{\partial})^{\#}$

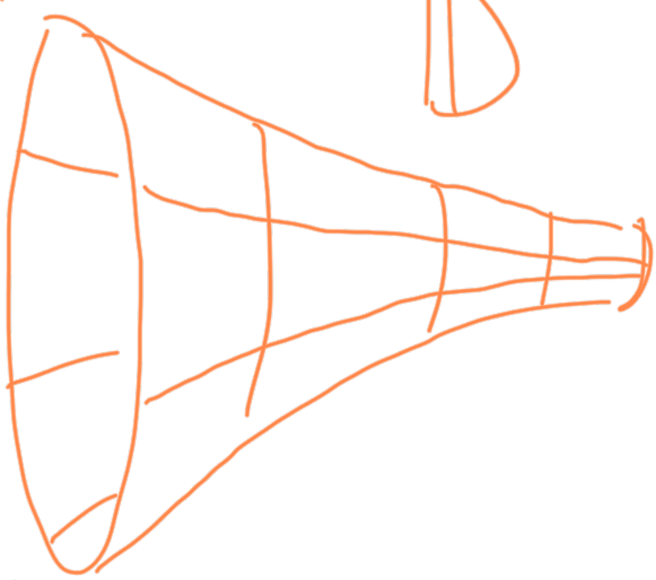
Hodge Theory:

$$H^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X)$$

"Kähler metrics are locally $\mathbb{C}$-Hessians"

Ex: Punctured Disk $\mathbb{D}^*$

$\tau = \ln(-\ln|z|^2)$



$$dd^c\tau = \frac{2i\, dz \wedge d\bar{z}}{|z|^2(\ln|z|^2)^2}$$

Constant Scalar Curvature Kähler Metric

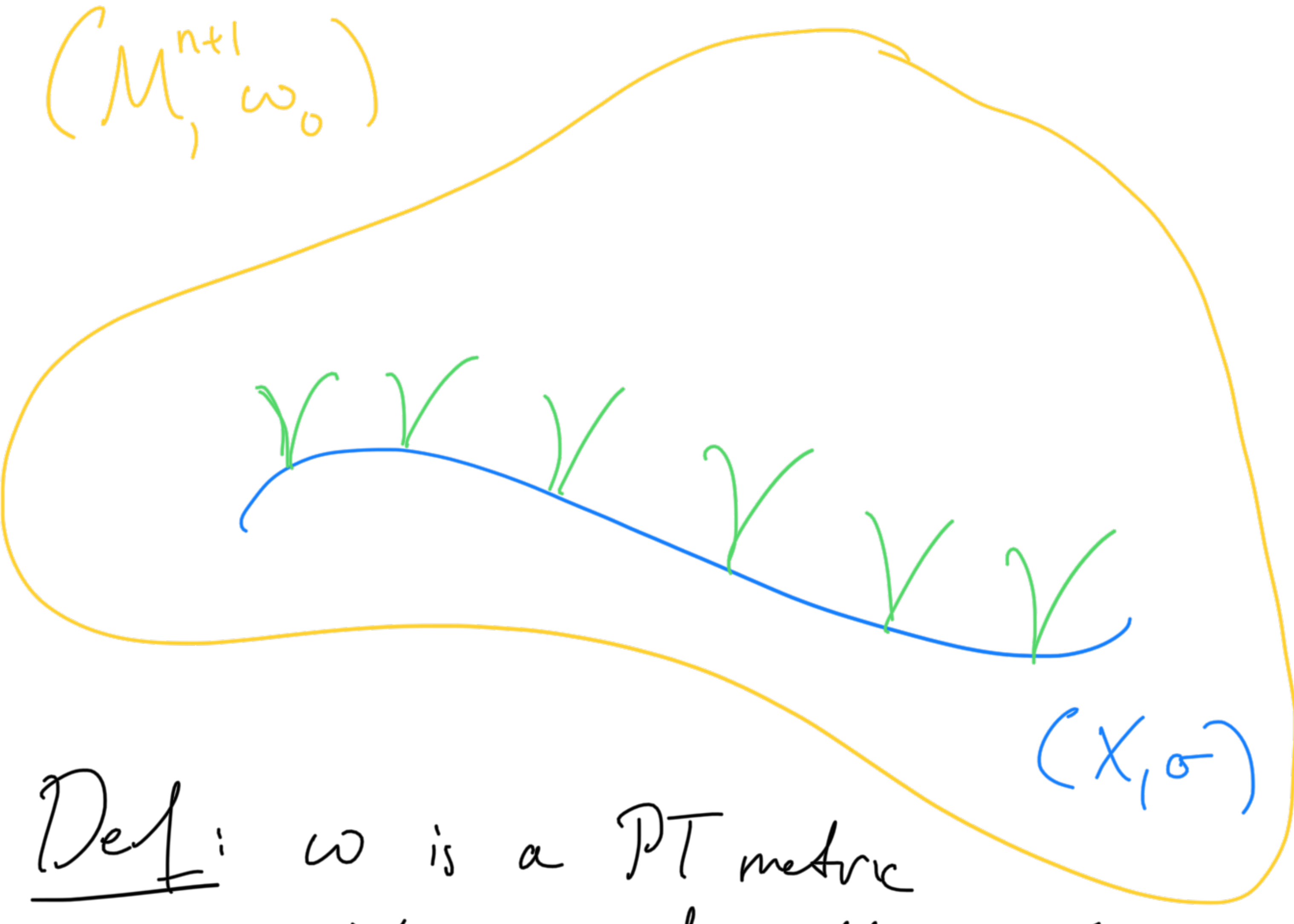
Ricci form $\rho = J \lrcorner \mathrm{Ric}$

$\mathrm{Scal}(\sigma)\sigma^n = 2n\rho\wedge\sigma^{n-1}$

$\rightarrow \mathrm{Scal}(dd^c\tau) = -1$

Kähler Metrics of Poincaré-Type

(Auvray, The space of Poincaré-type Kähler metrics on the complement of a divisor)



<u>Def</u>: ω is a PT metric on $M \setminus X$ means for all $p \in X$ $\exists$ holo chart $\mathcal{U}$ defining X (i.e. $X \cap \mathcal{U} = \{z = 0\}$) s.t. on $\mathcal{U}$, ω is quasi-isomorphic at all orders to

$$\frac{2i\,dz \wedge d\bar{z}}{|z|^2(\ln|z|^2)^2} + \omega_0|_X .$$

Moreover, $\omega \in [\omega_0]$ means $\exists \phi$ in $\mathcal{O}(\ln(-\ln|z|))$ with bounded derivatives such that $\omega = \omega_0 + dd^c \phi$

Recall such ϕ are "distortion potentials"

Note: PT metrics always exist

<u>Pf</u>: Let s be a defining section for the line bundle $\mathcal{O}_M[X]$ equipped with an Hermitian metric $\|\cdot\|$ small enough so that $\ln(-\ln\|s\|^2)$ is a real valued function on $M \setminus X$. Then $\omega_0 + dd^c \ln(-\ln\|s\|^2)$ is PT. ∎

<u>Rmks</u> Complete, finite volume, topological mean scalar curvature

Why generalize?

Quick detour to the Calabi Program...

$$\mathcal{C}a(\varphi) = \int_X \mathrm{Scal}(\omega + dd^c\varphi)^2 (\omega + dd^c\varphi)^n$$

Minimizers: extremal "extK"

Satisfy $\bar{\partial}\,\mathrm{Scal}(\omega + dd^c\varphi) = 0$

$$\Updownarrow$$

$$\mathcal{L}_{\mathrm{grad}\,\mathrm{Scal}(\omega + dd^c\varphi)} J = 0$$

KE $\subset$ cscK $\subset$ extK

Highlights intimacy between extK metrics and $\mathrm{Aut}_0(X)$.

<u>Thm</u> (LeBrun-Simanca 94)

For a compact Kähler manifold, the set of extremal Kähler classes is open in the Kähler cone.

Auvray provides an obstruction:

Thm (Auvray, '17): Assume there exists an extK (resp. cscK) PT metric $\omega \in [\omega_0]$ on $M \setminus X$. Then $\exists$ extK (resp. cscK) metric on X in $[\omega_0|_X]$.

Additional Fact:

Thm (Sektnan, '18): Let Z be the extremal v.f. for a PT metric on $M \setminus X$. Then Z extends holomorphically across X to all of M

Consider: $\mathrm{Aut}_0(M) = 1$

$\mathrm{Aut}_0(X) \neq 1$ $\left(\begin{array}{l} \text{or } h_M = 0 \\ \text{but } h_X \neq 0 \end{array} \right)$

$\lfloor X$,

Starting with $\omega \in [\omega_0]$ cscK PT on $M \setminus X$, for some $\eta \in \mathcal{H}^{1,1}(M)$, the class $[(\omega_0 + \eta)|_X]$ may not admit cscK metrics. If so, $\omega + \eta$ cannot be cscK.

Half-example: Multiple blowup of $(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \times X$.

Analytical issue is $\ker \mathcal{D}_\omega^* \mathcal{D}_\omega$

- Compact case is self adjoint, but linear theory is more complicated in PT setting

Solution? Add in some distortion potentials which ruin the PT asymptotics but work in linearization.

Matsushima
Lichnerowicz $\quad H^0(X, T_X)$
$$= \mathfrak{a} \oplus \mathfrak{h} \oplus J\mathfrak{h}$$

$\mathfrak{a}$: parallel fields (no zeroes)

$J\mathfrak{h}$: Hamiltonian w.r.t. σ

$\mathfrak{h}$ = gradient fields (with holomorphy potentials)

Take $V = \operatorname{grad} f_V$ with flow F_{tV}. Define ψ_{tV} according to

$$\frac{\partial}{\partial t} \psi_{tV} = F_{tV}^* f_V, \quad \psi_0 = 0$$

Note: $\quad dd^c \psi_{tV} = F_{tV}^* \sigma - \sigma$

Aubry's obstruction was local, so our construction will be local on

$N = \mathbb{D} \times X \quad \check{N} = \mathbb{D}^* \times X$

$$\nu : \check{N} \longrightarrow X$$

Def: Given $V \in \mathfrak{h}$, the gnarl associated to (V, σ) is the function

$$\Psi_V = (\tau \times \nu)^* \psi_{tV}$$

Set $\omega_V = \omega + dd^c \Psi_V$

Facts:

→ $(\check{N}, dd^c \tau + \nu^* \sigma)$ is $\overset{\text{cscK}}{\text{PT}}$

→ For small V, $\omega_V = dd^c \tau + \nu^* \tilde{\sigma} + dd^c \Psi_V$ is a complete metric on $\check{N}$ for any $\tilde{\sigma}$ on X

→ $F_{TV}^* \psi_{tV} = \psi_{(T+t)V} - \psi_{TV}$

→ F_{tV} extends to a family of isometries ... so that it is for ω_V ... it is sufficient to find $\varphi \in C^\infty(X)$ s.t.

$\omega_V + dd^c \nu^* \varphi$ has scalar curvature which is constant along a single slice $\{z_0\} \times X \subset \check{N}$

N.b. NOT the restriction metric

→ For every $k > 0$ $\exists\, \varepsilon_k >$ s.t.

$$\|\Psi_V\| \lesssim e^{\varepsilon_k \tau}$$

→ The linearization of $V \mapsto \Psi_V$ at the origin is $W \mapsto f_W$

Thm (A, '21): $\exists$ nbhd $\mathcal{U}$ of 0 in $\mathcal{H}^{1,1}(X)$ s.t. for every $\eta \in \mathcal{U}$, $\exists$ vector field $V_\eta \in H^0(X, T_X)$ and a cscK gravitized PT metric $\tilde{\omega}_{V_\eta}$ associated to V_η and a metric in $[\omega + \eta]$ on $\check{N}$.

Idea of proof: Define

$$S^! : \eta \oplus \mathcal{H}^{1,1}(X) \oplus C^{4,\alpha}(X) \to C^{0,\alpha}(X)$$

via

$$(\nu, \eta, \varphi) \longmapsto \mathrm{Scal}(\omega_{\nu,\eta,\varphi})\Big|_{\{e^{-2}\}\times X}$$

$$\hookrightarrow dd^c\tau + \nu^*(\sigma+\eta) + dd^c(\nu^*\varphi + \Psi_\nu^{\eta,\varphi})$$

$\hookrightarrow$ gnarl associated to

$$\sigma + \eta + dd^c\varphi$$

$$\Psi_\nu^{\eta,\varphi} = \Psi_\nu^{\eta} + \nu^* F_{\tau\nu}^* \varphi + \nu^*\varphi$$

$$\Psi_\nu^{\eta} = \nu^* G_\sigma \Lambda_\sigma (F_{\tau\nu}^*(\sigma+\eta) - \sigma - \eta)$$

Linearization:

$$\frac{d}{du}\Big|_{u=0} S^!(u\nu, 0, u\varphi)$$

$$= 2\mathcal{D}^* \mathcal{D}(\nu^*(\varphi + \tau f_\nu))\Big|_{\{e^{-2}\}\times X}$$

Sektnan

$$\mathcal{D}_\omega^* \mathcal{D}_\omega (\nu^*(\varphi + \tau f_\nu))$$

$$= \nu^* \mathcal{D}_o^* \mathcal{D}_o \varphi + f_V + \Delta_o f_V$$

is invertible.

Extensions

Using global construction, can define the gravel on all of $M \setminus X$ with cutoff.

- Weighted spaces / v.f. extension choice
- Normal crossings divisors
- Genuine openness for these metrics
- Improved Sektnan blow-up
- Other flows for the gravel?

Calabi flow

$$\frac{\partial}{\partial t}\omega_t = dd^c \,\mathrm{Scal}(\omega_t)$$

$$\frac{\partial}{\partial t}\phi_t = \mathrm{Scal}(\omega_t) - \underline{S}$$

Thank You