A relative entropy for Ricci expanders

Joint work with Felix Schulze (Warwick University)

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Parabolic version of [Anderson, Bando-Kasue-Nakajima]'s problem on Einstein metrics

Main definitions

Definition (Time-dependent definition)

An expanding gradient Ricci soliton is an immortal solution $(g(t))_{t>0}$ *to the Ricci flow such that*

$$g(t) = t\varphi_t^* g, \quad \partial_t \varphi_t = -t^{-1}\nabla^g f \circ \varphi_t, \quad t > 0, \quad \varphi_t\big|_{t=1} = \mathrm{Id}_M,$$

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If M is closed and $(M^n, g, \nabla^g f)$ is an E.G.S. then $\nabla^g f = 0$.

Main asymptotically conical examples

1. The Gaussian expanding gradient Ricci soliton $(\mathbb{R}^n, g_{eucl}, \frac{1}{2}r\partial_r)$, $\varphi_t(x) = \frac{x}{\sqrt{t}}$, $t > 0$, and its Kähler counterpart: $(\mathbb{C}^n, i\partial\overline{\partial}|\cdot|^2, r\partial_r)$.

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   $p = 1$: ALE metrics with exponential rate and $\mathrm{R}(g_{k,1}) > 0$.

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In the setting of Ricci shrinkers, the tangent cone at infinity determines the soliton metric: [Kotschwarr-Wang, 13']

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with asymptotic cone (C_0, g_0).

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The Kähler case

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- $d\pi(X) = r\partial_r$ and $\nabla^{g_0,k}(\pi_* g - g_0 - \mathrm{Ric}(g_0)) = O(r^{-4-k})$ for all $k \geq 0$.

Theorem (Conlon-D', 16', Conlon-D'-Sun, 19')

Let $(C_0, J_0, g_0, r\partial_r)$ be a Kähler cone. Then the following assertions are equivalent:

1. *There exists a unique complete expanding gradient Kähler-Ricci soliton (M^{2n}, J, g, X) such that*
$$\nabla^{g,k}\,\mathrm{Rm}(g) = O(r^{-2-k}), \quad k \geq 0,$$
with asymptotic cone (C_0, g_0).
2. *C_0 admits a smooth resolution $\pi : M \to C_0$ such that the canonical line bundle $K_M|_E$ is π-ample, i.e. $c_1(K_M|_E) > 0$, if $\pi^{-1}(\{0\}) =: E$ denotes the exceptional set.*

Such a resolution $\pi : M \to C_0$ satisfies:

- $d\pi(X) = r\partial_r$ and $\nabla^{g_0,k}(\pi_* g - g_0 - \mathrm{Ric}(g_0)) = O(r^{-4-k})$ for all $k \geq 0$.
- the torus action on C_0 generated by $J_0(r\partial_r)$ extends to a holomorphic isometric action on (M, J, g).

Unique continuation results

Theorem (D'-Schulze, 21')

Let $(M^n, g_i, \nabla^{g_i} f_i)$, $i = 1, 2$ *be two expanding gradient Ricci solitons coming out of the same cone* $(C(S), g_C := dr^2 + r^2 g_S, \frac{r}{2}\partial_r)$ *over a smooth link* (S, g_S).

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Moreover, g_1 *and* g_2 *coincide pointwise outside a compact set if and only if their associated trace at infinity vanishes, i.e.*

$$\mathrm{tr}_\infty\left(r^n e^{\frac{r^2}{4}}(g_1 - g_2)\right) \equiv 0\,.$$

Some ideas of the proof

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$$\Delta_{g_1} h + \nabla^{g_1}_{\nabla^{g_1} f_1} h = R[h] + \mathcal{L}_{\mathcal{B}(h)}(g_1), \quad \mathcal{B}(h) := \mathrm{div}_{g_1} h - \frac{1}{2}\nabla^{g_1} \mathrm{tr}_{g_1} h.$$

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- (Kotschwar, 17$'$) If $h(t) := t\varphi_t^*(g_2 - g_1)$, $\mathcal{B}(h(t))$ satisfies an ODE, i.e.

$$\nabla^{g_1}_{\nabla^{g_1} f_1} \mathcal{B}(h) - \frac{\mathcal{B}(h)}{2} = R[h].$$

  Based on Bianchi identity: $\mathcal{B}(\mathrm{Ric}(g_i)) = 0$.

- Following (Bernstein, 17′): use of a frequency function associated to $h = g_2 - g_1$:

$$N(R) := R\frac{\int_{r\geq R}|\nabla\hat{h}|^2 r^{-2n}e^{-\frac{r^2}{4}}\,d\mu_g}{\int_{r=R}|\hat{h}|^2 r^{-2n}e^{-\frac{r^2}{4}}\,d\sigma_g}, \quad \hat{h} := r^n e^{\frac{r^2}{4}}h.$$

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if $B(R) := \int_{r=R} |\hat{h}|^2$,

$$\frac{d}{dR} R^{1-n} B(R) = O_\varepsilon(R^{-3+\varepsilon}) R^{1-n} B(R),$$

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Then the following limit exists for all $t > 0$ *and is constant in time:*

$$\mathcal{W}(g_2(t), g_1(t)) := \lim_{R \to +\infty} \left(\int_{f_2(t) \leq R} \frac{e^{f_2(t)}}{(4\pi t)^{\frac{n}{2}}} \, d\mu_{g_2(t)} - \int_{f_1(t) \leq R} \frac{e^{f_1(t)}}{(4\pi t)^{\frac{n}{2}}} \, d\mu_{g_1(t)} \right).$$

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Thanks !