A relative entropy for Ricci expanders

Joint work with Felix Schulze (Warwick University)
• Ricci flow on a closed manifold $M^n$, $n \geq 2$ (Hamilton, 82'):

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2 \text{Ric}(g(t)) \quad \text{on } M^n \times (0, T_{\text{max}}), \\ g(0) = g_0 \quad \text{on } M^n, \end{cases}$$

and its Kähler counterpart: $(M^{2n}, \omega(t))_{t \in (0, T_{\text{max}})}$, $\partial_t \omega = -\rho(\omega(t))$. 

Two features of the Ricci flow:

- Instantaneous regularization: $L^\infty$ perturbation of Euclidean space (Koch-Lamm, 11').
- Convergence to a canonical metric: Uniformisation of surfaces (Hamilton, 88', Chow, 91', Chen-Lu-Tian, 06'), Uniformisation of 3-manifolds with positive Ricci curvature (Hamilton, 82'), $n$-manifolds with 2-positive curvature operator (Böhm-Wilking, 06').
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Question

How can (expanding) solitons help to reverse this process?
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Question

*How can (expanding) solitons help to reverse this process?*

Parabolic version of [Anderson, Bando-Kasue-Nakajima]'s problem on Einstein metrics
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<th>Definition (Time-dependent definition)</th>
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An expanding gradient Ricci soliton is an immortal solution \((g(t))_{t>0}\) to the Ricci flow such that

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g(t) = t\varphi_t^* g, \quad \partial_t \varphi_t = -t^{-1} \nabla g f \circ \varphi_t, \quad t > 0, \quad \varphi_t \big|_{t=1} = \text{Id}_M,
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**Definition (Static definition)**

An expanding gradient Ricci soliton (E.G.S.) is a triple \((M^n, g, \nabla g f)\) satisfying:

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\text{Ric}(g) - \nabla g.f = -\frac{g}{2},
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Bakry-Émery tensor.

Joint work with Felix Schulze (Warwick University)
**Main definitions**

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An expanding gradient Kähler Ricci soliton is an immortal solution \((g(t))_{t>0}\) to the Kähler-Ricci flow such that

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If \( M \) is closed and \((M^n, g, \nabla g f)\) is an E.G.S. then \( \nabla g f = 0 \).

Joint work with Felix Schulze (Warwick University)
Main asymptotically conical examples

1. The Gaussian expanding gradient Ricci soliton \((\mathbb{R}^n, g_{\text{eucl}}, \frac{1}{2} r \partial_r)\), \(\varphi_t(x) = \frac{x}{\sqrt{t}}\), \(t > 0\), and its Kähler counterpart: \((\mathbb{C}^n, i \partial \bar{\partial} \cdot |^2, r \partial_r)\).
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\(p = 1\): ALE metrics with exponential rate and \(\text{R}(g_{k,1}) > 0\).

Joint work with Felix Schulze (Warwick University)
Main interest

- Generalization of (Kähler)-Einstein metrics.
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- Intuition on arbitrary immortal Type III solutions ($|Rm(g(t))|_{g(t)} \leq C/t$):

  Hamilton's Matrix Harnack estimate (Hamilton, 93′):
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  Candidates for smoothing isolated singularities out instantaneously.

  \[\Rightarrow\] produces a complete solution going through the singularities eventually.

  [Gianniotis-Schulze, 16′]: existence of complete solutions of the Ricci flow starting from closed manifolds with isolated singularities modeled on cones with non-negative curvature operator.

  [Kröncke-Vertman, 18′]: existence of the (DeTurck)-Ricci flow from closed manifolds with isolated singularities modelled on “strictly stable” Ricci flat cones.

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  Feldman-Ilmanen-Knopf examples
Main questions

1. Given a metric cone over a smooth Riemannian link, how many self-similar solutions come out of it?

2. Existence: Dirichlet problem at infinity.

3. Uniqueness: Given two self-similar solutions coming out of the same metric cone, under which (geometric) assumption are they isometric?

[Angenent-Knopf, 1999] There are Ricci flat cones (dim. \( \geq 5 \)) admitting non-isometric complete expanding self-similar evolutions.

In the setting of Ricci shrinkers, the tangent cone at infinity determines the soliton metric: [Kotschwar-Wang, 2013']

Joint work with Felix Schulze (Warwick University): A relative entropy for Ricci expanders
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2. Existence: Dirichlet problem at infinity.

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   Given two self-similar solutions coming out of the same metric cone, under which (geometric) assumption are they isometric?

   [Angenent-Knopf, 19′] There are Ricci flat cones (dim. \(\geq 5\)) admitting non-isometric complete expanding self-similar evolutions.

   In the setting of Ricci shrinkers, the tangent cone at infinity determines the soliton metric:

   [Kotschwarr-Wang, 13′]
Theorem (Conlon-D’, 16’, Conlon-D’-Sun, 19’)

Let \((C_0, J_0, g_0, r \partial_r)\) be a Kähler cone. Then the following assertions are equivalent:

1. There exists a unique complete expanding gradient Kähler-Ricci soliton \((M, J, g, X)\) such that
   \[
   \nabla g, k R_{m}(g) = O(r^{-2-k}),
   \]
   with asymptotic cone \((C_0, g_0)\).

2. \(C_0\) admits a smooth resolution \(\pi: M \to C_0\) such that the canonical line bundle
   \(K_M|_E\) is \(\pi\)-ample, i.e.
   \[
   c_1(K_M|_E) > 0,
   \]
   if \(E = \pi^{-1}(\{0\})\) denotes the exceptional set.

Such a resolution \(\pi: M \to C_0\) satisfies:

\[
\nabla g, k (\pi^* g - g_0 - \text{Ric}(g_0)) = O(r^{-4-k})
\]
for all \(k \geq 0\).

Joint work with Felix Schulze (Warwick University)
The Kähler case

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\[
\nabla^{g,k} \mathrm{Rm}(g) = O(r^{-2-\varepsilon}), \quad k \geq 0,
\]

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Joint work with Felix Schulze (Warwick University)

A relative entropy for Ricci expanders
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- the torus action on \(C_0\) generated by \(J_0(r\partial_r)\) extends to a holomorphic isometric action on \((M, J, g)\).

Joint work with Felix Schulze (Warwick University)
Theorem (D’-Schulze, 21’)

Let \((M^n, g_i, \nabla^g_i f_i), i = 1, 2\) be two expanding gradient Ricci solitons coming out of the same cone \((C(S), g_C := dr^2 + r^2 g_S, \frac{r}{2} \partial_r)\) over a smooth link \((S, g_S)\).
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Then the trace at infinity

\[
\lim_{r \to +\infty} r^n e^{\frac{r^2}{4}} (g_1 - g_2) =: \text{tr}_\infty \left( r^n e^{\frac{r^2}{4}} (g_1 - g_2) \right)
\]

exists in the \(L^2_{\text{loc}}(C(S))\)-topology, it preserves the radial vector field \(\partial_r\) and its tangential part is divergence free with respect to the metric on the link in the weak sense.

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Moreover, \(g_1\) and \(g_2\) coincide pointwise outside a compact set if and only if their associated trace at infinity vanishes, i.e.

\[
\text{tr}_\infty \left( r^n e^{\frac{r^2}{4}} (g_1 - g_2) \right) \equiv 0.
\]
Some ideas of the proof

Previous unique continuation result when the cone is Ricci flat: [D', 16']
Some ideas of the proof

Previous unique continuation result when the cone is Ricci flat: [D’, 16’]

Here, the convergence holds in the pointwise sense: proof based on Carleman estimates in the spirit of (Donnelly, 99’)

Assume \( \nabla g_2 f_2 = \nabla g_1 f_1 \) at infinity and define \( h := g_2 - g_1 \):

\[
\Delta g_1 h + \nabla g_1 \nabla g_1 f_1 h = R[h] + L_B(h)(g_1),
\]

\( B(h) := \text{div} g_1 h - \frac{1}{2} \nabla g_1 \text{tr} g_1 h \).

\[
\Rightarrow \text{degenerate elliptic equation.}
\]

\[
\Delta g_1 + \nabla g_1 \nabla g_1 f_1 \text{is asymptotic to } (\Delta g_{\text{cone}} + \frac{1}{2} r \partial r) \text{ and conjugate to a harmonic oscillator.}
\]

(Kotschwar, 17’)

If \( h(t) = t \phi \ast t(g_2 - g_1) \), \( B(h(t)) \) satisfies an ODE, i.e.

\[
\nabla g_1 \nabla g_1 f_1 B(h) - B(h)^2 = R[h].
\]

Based on Bianchi identity:

\[
B(Ric(g_i)) = 0.
\]
Some ideas of the proof

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Reminiscent of (Mazzeo, 91’) and (Biquard, 08’) for conformally compact Einstein metrics

A priori, $g_i - g_{cone} = \text{Ric}(g_{cone}) + O(r^{-4})$ only $\Rightarrow g^2 - g^1 = O(r^{-4}).$

Assume $\nabla g^2 f^2 = \nabla g^1 f^1$ at infinity and define $h := g^2 - g^1$:

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\[ \Delta g^1 + \nabla g^1 \nabla g^1 f_1 \text{is asymptotic to } \left( \Delta g^\text{cone} + \frac{1}{2} r \partial_r \right) \text{ and conjugate to a harmonic oscillator.} \]

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- Assume $\nabla^{g_2} f_2 = \nabla^{g_1} f_1$ at infinity and define $h := g_2 - g_1$:

  $$\Delta_{g_1} h + \nabla_{\nabla^{g_1} f_1} h = R[h] + L_{\mathcal{B}(h)}(g_1), \quad \mathcal{B}(h) := \text{div}_{g_1} h - \frac{1}{2} \nabla^{g_1} \text{tr}_{g_1} h.$$

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$\Rightarrow$ degenerate elliptic equation.

• $\Delta_{g_1} + \nabla^{g_1}_{\nabla^{g_1} f_1}$ is asymptotic to $(\Delta_{g_{\text{cone}}} + \frac{1}{2} r \partial_r)$ and conjugate to a harmonic oscillator.
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- (Kotschwar, 17’) If \(h(t) := t \varphi_t^*(g_2 - g_1), \mathcal{B}(h(t))\) satisfies an ODE, i.e.

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Based on Bianchi identity: \(\mathcal{B}(\text{Ric}(g_i)) = 0\).
Some ideas of the proof

- Following (Bernstein, 17'): use of a frequency function associated to $h = g_2 - g_1$:

$$N(R) := R \frac{\int_{r \geq R} |\nabla \hat{h}|^2 r^{-2n} e^{-\frac{r^2}{4}} \, d\mu_g}{\int_{r=R} |\hat{h}|^2 r^{-2n} e^{-\frac{r^2}{4}} \, d\sigma_g}, \quad \hat{h} := r^n e^{\frac{r^2}{4}} h.$$
Some ideas of the proof

- Following (Bernstein, 17′): use of a frequency function associated to \( h = g_2 - g_1 \):

\[
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- Based on ideas of (Almgren, 70’s) and (Garofalo-Lin, 90’s) on unique continuation results for the laplacian.
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- Here we get for \( \varepsilon \in (0, 1) \): \( N(R) = O_\varepsilon(R^{-2+\varepsilon}) \) and \( N \) is almost decreasing.
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\[
\text{if } B(R) := \int_{r = R} |\hat{h}|^2,
\]

\[
\frac{d}{dR} R^{1-n} B(R) = O_\varepsilon (R^{-3+\varepsilon}) R^{1-n} B(R),
\]

\[
\int_{r \geq R} |\nabla \hat{h}|^2 r^{-2n} e^{-\frac{r^2}{4}} \, d\mu_g = N(R)B(R)R^{-2n-1}e^{-\frac{R^2}{4}}.
\]
Application: existence of a relative entropy

Based on [Feldman-Ilmanen-Ni, 05'],

**Theorem (D’-Schulze, 21')**

Let $\left( M^n, g_i, \nabla g_i, f_i \right)$, $i = 1, 2$ be two complete expanding gradient Ricci solitons coming out of the same cone $\left( C(\mathbb{S}^n), g_C := dr^2 + r^2 g_{\mathbb{S}^n}, r^2 \partial_r \right)$ over a smooth link $\left( \mathbb{S}^n, g_{\mathbb{S}^n} \right)$.

Then the following limit exists for all $t > 0$ and is constant in time:

$$W (\left. g_2 \right|_{t}, g_1 \right|_{t}) := \lim_{R \to +\infty} \left( \int f_2 (t) \leq R e^{f_2 (t)} (4\pi t)^{-\frac{n}{2}} d\mu_{g_2 (t)} - \int f_1 (t) \leq R e^{f_1 (t)} (4\pi t)^{-\frac{n}{2}} d\mu_{g_1 (t)} \right).$$

Relative entropy developed for the mean curvature flow: [D’-Schulze, 19′], [Bernstein-Wang, 19′].

Improper integral:

$$\left( \partial_t + \Delta g(t) - R g(t) \right) e^{f(t)} (4\pi t)^{-\frac{n}{2}} = 0,$$

$$f(t) := \phi^* t f.$$

Hope: generic uniqueness of expanders with zero relative entropy coming out of a cone.

True for expanders of the mean curvature flow: [D’-Schulze, 19′].

Joint work with Felix Schulze (Warwick University)
Application: existence of a relative entropy

Based on [Feldman-Ilmanen-Ni, 05'],

**Theorem (D'-Schulze, 21')**

Let \((M^n, g_i, \nabla g_i f_i), i = 1, 2\) be two complete expanding gradient Ricci solitons coming out of the same cone \((C(S), g_C := dr^2 + r^2 g_S, \frac{r}{2} \partial_r)\) over a smooth link \((S, g_S)\).
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**Theorem (D’-Schulze, 21’)**

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\[
\mathcal{W}(g_2(t), g_1(t)) := \lim_{R \to +\infty} \left( \int_{f_2(t) \leq R} \frac{e^{f_2(t)}}{(4\pi t)^{n/2}} d\mu_{g_2(t)} - \int_{f_1(t) \leq R} \frac{e^{f_1(t)}}{(4\pi t)^{n/2}} d\mu_{g_1(t)} \right).
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\mathcal{W}(g_2(t), g_1(t)) := \lim_{R \to +\infty} \left( \int_{f_2(t) \leq R} \frac{e^{f_2(t)}}{(4\pi t)^{\frac{n}{2}}} \, d\mu_{g_2}(t) - \int_{f_1(t) \leq R} \frac{e^{f_1(t)}}{(4\pi t)^{\frac{n}{2}}} \, d\mu_{g_1}(t) \right).
\]

Relative entropy developed for the mean curvature flow: [D'-Schulze, 19'], [Bernstein-Wang, 19'].

Improper integral:

\[
(\partial_t + \Delta_{g(t)} - R_{g(t)}) \frac{e^f(t)}{(4\pi t)^{\frac{n}{2}}} = 0, \quad f(t) := \varphi^*_t f.
\]
Application: existence of a relative entropy

Based on [Feldman-Ilmanen-Ni, 05'],

**Theorem (D'-Schulze, 21')**

Let \((M^n, g_i, \nabla g_i f_i)\), \(i = 1, 2\) be two complete expanding gradient Ricci solitons coming out of the same cone \((C(S), g_C := dr^2 + r^2 g_S, \frac{r}{2} \partial_r)\) over a smooth link \((S, g_S)\).

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Hope: generic uniqueness of expanders with zero relative entropy coming out of a cone.

Joint work with Felix Schulze (Warwick University)
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Hope: generic uniqueness of expanders with zero relative entropy coming out of a cone.

True for expanders of the mean curvature flow: [D'-Schulze, 19']
Thanks!