Ampleness of vector bundles and canonical metrics

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Schneider-Tancredi

Theorem (Schneider and Tancredi, Manuscripta Mathematica, 1985)

Let E be a rank two holomorphic vector bundle over a compact complex surface X. Assume that $c_1(E) > 0$ and that E is semistable with respect to $\det(E)$. Suppose $E\big|_C$ is ample for every closed curve $C \subset X$, and

$$(c_1(E)^2 - 2c_2(E)).X > 0, \quad c_2(E).X > 0.$$

Then E is ample.

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- Unlike line bundles, there is no unique positivity notion for the curvature F.
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Theorem (Lübke, Indagationes Mathematicae, 1991)

Let (E, h) be a holomorphic Hermitian rank r vector bundle over a compact Kähler manifold (X, ω). Suppose $F_h \wedge \omega^{n-1} = -\sqrt{-1}\lambda\omega^n$, where F_h is the curvature of the Chern connection of h and $\lambda > 0$ is a constant. Assume that

$$c_1(E, h) = \frac{r\lambda}{2\pi}\omega.$$

Suppose there exists a positive function ψ such that either of the following holds:

1. $n = 2$ *and* $c_1^2(E, h) - \frac{2r(r-1)}{r^2-2r+2}c_2(E, h) = \psi\omega^2$, *or*
2. $r = 2$ *and* $c_1^2(E, h) - \frac{4(n-1)^2}{n^2-2n+2}c_2(E, h) = \psi\omega^2$.

Then h is GP.

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$$\left(\sqrt{-1}F_{h_t} - \frac{\sqrt{-1}}{r}\mathrm{tr}F_{h_t}\right)\omega_0^{n-1} = -\epsilon\frac{(\det h_0)^\mu}{(\det h)^\mu}\ln\left(\frac{hh_0^{-1}}{\det(hh_0^{-1})^{1/r}}\right)\omega_0^n, \quad (4)$$

where h_0 is a smooth background Hermitian metric, $\mu, \lambda \geq 0$ are fixed constants, $\alpha > 0$ is a large enough constant so that $\Theta_{h_0} + \alpha\omega$ is dual-Nakano positively curved, and $f_t > 0$ are smooth positive functions.

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We prove (P., C.R.Acad. Sci. '21) the following theorem.

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Let E be an ω_0-stable rank$-r$ holomorphic bundle on X. Let H_0 be a Hermitian-Einstein metric on E with respect to ω_0, that is, $\sqrt{-1}F_{H_0}\omega_0^{n-1} = \lambda\omega_0^n$. Let h be a smooth metric on E solving the following cushioned Hermitian-Einstein equation for given parameters $\epsilon \geq 0, \mu \geq 0$.

$$\left(\sqrt{-1}F_h - \frac{\sqrt{-1}}{r}\mathrm{tr}F_h\right)\omega_0^{n-1} = -\epsilon\frac{(\det H_0)^\mu}{(\det h)^\mu}\ln\left(\frac{hH_0^{-1}}{\det(hH_0^{-1})^{1/r}}\right)\omega_0^n, \tag{5}$$

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$$\det_{TX \otimes E^*} \left(\sqrt{-1} F_{h_t} + (1-t)\alpha\omega_0 \otimes I_{E^*} \right)^{1/r} = f_t \frac{(\det h_0)^{\lambda}}{(\det h_t)^{\lambda}} \omega_0^n, \qquad (6)$$

$$\left(\sqrt{-1} F_{h_t} - \frac{\sqrt{-1}}{r} \mathrm{tr} F_{h_t} \right) \omega_t^{n-1} = -\epsilon \frac{(\det h_0)^{\mu}}{(\det h)^{\mu}} \ln \left(\frac{h h_0^{-1}}{\det(h h_0^{-1})^{1/r}} \right) \omega_0^n. \qquad (7)$$

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A Kobayashi-Hitchin-Donaldson-Uhlenbeck-Yau correspondence for the vortex case

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$$i\Theta_h = (1 - |\phi|_h^2)\frac{(2(r_2(r_1+1) + r_1(r_2+1)))\omega_\Sigma + i\nabla_h^{1,0}\phi \wedge \nabla^{0,1}\phi^{\dagger_h}}{(2r_2 + |\phi|_h^2)(2 + 2r_2 - |\phi|_h^2)}.$$

Moreover,

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Moreover, if a solution h to **??** *satisfying $|\phi|_h^2 \leq 1$ exists, then it is unique.*

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$$= \eta \frac{\int_M 2\mathrm{ch}_2(E \otimes L^k)}{\int_M r\eta} Id = k^2 \eta Id + k \frac{\int_M 2c_1(E)c_1(L)}{\int_M r\eta} \eta Id + \frac{\int_M 2\mathrm{ch}_2(E)}{\int_M r\eta} \eta$$

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Thank you