A Chern-Calabi flow on Hermitian Manifolds

Xi Sisi Shen

Department of Mathematics
Columbia University

UQAM Geometry and Topology Seminar

Outline of Talk

Background

In the Kähler setting

- The Mabuchi energy
- Convergence of the Calabi flow

Generalizing to the non-Kähler setting

- A Chern-Calabi flow for Hermitian metrics
- Outline of the proof of the estimates

Setting

Let (X^n, J, ω) be a compact complex manifold of complex dimension n and ω a Hermitian metric on X.

Locally, $\omega = \sqrt{-1}\sum_{i=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is a real $(1,1)$-form and $(g_{i\bar{j}})$ is a positive-definite Hermitian matrix.

ω is Kähler if and only if $d\omega = 0$.

Setting

Let (X^n, J, ω) be a compact complex manifold of complex dimension n and ω a Hermitian metric on X.

Locally, $\omega = \sqrt{-1}\sum_{i=1}^n g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is a real $(1,1)$-form and $(g_{i\bar{j}})$ is a positive-definite Hermitian matrix.

ω is Kähler if and only if $d\omega = 0$.

Setting

Let (X^n, J, ω) be a compact complex manifold of complex dimension n and ω a Hermitian metric on X.

Locally, $\omega = \sqrt{-1}\sum_{i=1}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is a real $(1,1)$-form and $(g_{i\bar{j}})$ is a positive-definite Hermitian matrix.

ω is Kähler if and only if $d\omega = 0$.

Notions of curvature

- The *Chern-Ricci form* is a $(1,1)$-form given by $\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det g$
- The *first Chern class* $c_1(X) = [\mathrm{Ric}(\omega)]$
- The *Chern scalar curvature* $\mathrm{R}(\omega) = \mathrm{tr}_\omega \mathrm{Ric}(\omega)$
- When the metric is Kähler, the Chern and Levi-Civita connections agree.

Notions of curvature

- The *Chern-Ricci form* is a $(1,1)$-form given by $\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial} \log \det g$
- The *first Chern class* $c_1(X) = [\mathrm{Ric}(\omega)]$
- The *Chern scalar curvature* $\mathrm{R}(\omega) = \mathrm{tr}_\omega \mathrm{Ric}(\omega)$
- When the metric is Kähler, the Chern and Levi-Civita connections agree.

Notions of curvature

- The *Chern-Ricci form* is a $(1, 1)$-form given by $\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial} \log \det g$
- The *first Chern class* $c_1(X) = [\mathrm{Ric}(\omega)]$
- The *Chern scalar curvature* $\mathrm{R}(\omega) = \mathrm{tr}_\omega \mathrm{Ric}(\omega)$
- When the metric is Kähler, the Chern and Levi-Civita connections agree.

Notions of curvature

- The *Chern-Ricci form* is a $(1,1)$-form given by $\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial} \log \det g$
- The *first Chern class* $c_1(X) = [\mathrm{Ric}(\omega)]$
- The *Chern scalar curvature* $\mathrm{R}(\omega) = \mathrm{tr}_\omega \mathrm{Ric}(\omega)$
- When the metric is Kähler, the Chern and Levi-Civita connections agree.

The Calabi functional

The Calabi functional is given by

$$\mathrm{Cal}(\omega) = \int_X \mathrm{R}(\omega)^2 \omega^n = \int_X (\mathrm{R}(\omega) - \underline{\mathrm{R}})^2 \omega^n + \int_X \underline{\mathrm{R}}^2 \omega^n$$

where $\underline{\mathrm{R}} = \frac{\int_X \mathrm{R}(\omega)\omega^n}{\int_X \omega^n} = \frac{n \int_X \mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\int_X \omega^n} = \frac{2\pi n c_1(X) \cdot [\omega]^{n-1}}{[\omega]^n}$ is the average scalar curvature and is an invariant of $[\omega]$.

- Extremal metrics are defined as critical points of the Calabi functional
- CscK metrics satisfy $\mathrm{R}(\omega) = \underline{\mathrm{R}}$ and are minimizers of the Calabi functional

The Calabi functional

The Calabi functional is given by

$$\mathrm{Cal}(\omega) = \int_X \mathrm{R}(\omega)^2 \omega^n = \int_X (\mathrm{R}(\omega) - \underline{\mathrm{R}})^2 \omega^n + \int_X \underline{\mathrm{R}}^2 \omega^n$$

where $\underline{\mathrm{R}} = \frac{\int_X \mathrm{R}(\omega)\omega^n}{\int_X \omega^n} = \frac{n \int_X \mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\int_X \omega^n} = \frac{2\pi n c_1(X) \cdot [\omega]^{n-1}}{[\omega]^n}$ is the average scalar curvature and is an invariant of $[\omega]$.

- Extremal metrics are defined as critical points of the Calabi functional
- CscK metrics satisfy $\mathrm{R}(\omega) = \underline{\mathrm{R}}$ and are minimizers of the Calabi functional

The Calabi functional

The Calabi functional is given by

$$\mathrm{Cal}(\omega) = \int_X \mathrm{R}(\omega)^2 \omega^n = \int_X (\mathrm{R}(\omega) - \underline{\mathrm{R}})^2 \omega^n + \int_X \underline{\mathrm{R}}^2 \omega^n$$

where $\underline{\mathrm{R}} = \frac{\int_X \mathrm{R}(\omega)\omega^n}{\int_X \omega^n} = \frac{n \int_X \mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\int_X \omega^n} = \frac{2\pi n c_1(X) \cdot [\omega]^{n-1}}{[\omega]^n}$ is the average scalar curvature and is an invariant of $[\omega]$.

- Extremal metrics are defined as critical points of the Calabi functional
- CscK metrics satisfy $\mathrm{R}(\omega) = \underline{\mathrm{R}}$ and are minimizers of the Calabi functional

The Calabi functional

The Calabi functional is given by

$$\mathrm{Cal}(\omega) = \int_X \mathrm{R}(\omega)^2 \omega^n = \int_X (\mathrm{R}(\omega) - \underline{\mathrm{R}})^2 \omega^n + \int_X \underline{\mathrm{R}}^2 \omega^n$$

where $\underline{\mathrm{R}} = \frac{\int_X \mathrm{R}(\omega)\omega^n}{\int_X \omega^n} = \frac{n \int_X \mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\int_X \omega^n} = \frac{2\pi n c_1(X) \cdot [\omega]^{n-1}}{[\omega]^n}$ is the average scalar curvature and is an invariant of $[\omega]$.

- Extremal metrics are defined as critical points of the Calabi functional
- CscK metrics satisfy $\mathrm{R}(\omega) = \underline{\mathrm{R}}$ and are minimizers of the Calabi functional

The Calabi functional

The Calabi functional is given by

$$\mathrm{Cal}(\omega) = \int_X \mathrm{R}(\omega)^2 \omega^n = \int_X (\mathrm{R}(\omega) - \underline{\mathrm{R}})^2 \omega^n + \int_X \underline{\mathrm{R}}^2 \omega^n$$

where $\underline{\mathrm{R}} = \frac{\int_X \mathrm{R}(\omega)\omega^n}{\int_X \omega^n} = \frac{n \int_X \mathrm{Ric}(\omega) \wedge \omega^{n-1}}{\int_X \omega^n} = \frac{2\pi n c_1(X) \cdot [\omega]^{n-1}}{[\omega]^n}$ is the average scalar curvature and is an invariant of $[\omega]$.

- Extremal metrics are defined as critical points of the Calabi functional
- CscK metrics satisfy $\mathrm{R}(\omega) = \underline{\mathrm{R}}$ and are minimizers of the Calabi functional

Examples of cscK metrics

- Kähler-Einstein metrics are those that satisfy $\mathrm{Ric}(\omega) = \lambda\omega$, where up to rescaling, $\lambda \in \{-1, 0, 1\}$.
  - Fubini-Study metric on $\mathbb{CP}^n$ ($\lambda = 1$)
  - Calabi-Yau metrics ($\lambda = 0$)
  - Hyperbolic metric on Riemann surfaces of genus ≥ 2 ($\lambda = -1$)
- Kähler-Einstein $\Rightarrow$ CscK $\Rightarrow$ Extremal

Yau '78

For $c_1(X) = 0$, there exists a unique Ricci-flat metric in any given Kähler class.

Examples of cscK metrics

- Kähler-Einstein metrics are those that satisfy $\mathrm{Ric}(\omega) = \lambda\omega$, where up to rescaling, $\lambda \in \{-1, 0, 1\}$.
  - Fubini-Study metric on $\mathbb{CP}^n$ ($\lambda = 1$)
  - Calabi-Yau metrics ($\lambda = 0$)
  - Hyperbolic metric on Riemann surfaces of genus ≥ 2 ($\lambda = -1$)

- Kähler-Einstein $\Rightarrow$ CscK $\Rightarrow$ Extremal

Yau '78

For $c_1(X) = 0$, there exists a unique Ricci-flat metric in any given Kähler class.

Examples of cscK metrics

- Kähler-Einstein metrics are those that satisfy $\mathrm{Ric}(\omega) = \lambda\omega$, where up to rescaling, $\lambda \in \{-1, 0, 1\}$.
  - Fubini-Study metric on $\mathbb{CP}^n$ ($\lambda = 1$)
  - Calabi-Yau metrics ($\lambda = 0$)
  - Hyperbolic metric on Riemann surfaces of genus ≥ 2 ($\lambda = -1$)

- Kähler-Einstein $\Rightarrow$ CscK $\Rightarrow$ Extremal

Yau '78

For $c_1(X) = 0$, there exists a unique Ricci-flat metric in any given Kähler class.

Examples of cscK metrics

- Kähler-Einstein metrics are those that satisfy $\text{Ric}(\omega) = \lambda\omega$, where up to rescaling, $\lambda \in \{-1, 0, 1\}$.
  - Fubini-Study metric on $\mathbb{CP}^n$ ($\lambda = 1$)
  - Calabi-Yau metrics ($\lambda = 0$)
  - Hyperbolic metric on Riemann surfaces of genus ≥ 2 ($\lambda = -1$)
- Kähler-Einstein $\Rightarrow$ CscK $\Rightarrow$ Extremal

Yau '78

For $c_1(X) = 0$, there exists a unique Ricci-flat metric in any given Kähler class.

Existence of cscK metrics

The cscK equation for $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ can be written as:

$$F = \log \frac{\omega_\varphi^n}{\omega^n}$$
$$\Delta_\varphi F = -\underline{R} + \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega).$$

Chen-Cheng prove a priori estimates under the assumption of bounded entropy $\mathrm{Ent} = \int_X \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n$:

Chen-Cheng '17

If (X, ω_φ) is a cscK metric, where $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$, then $||\varphi||_{C^k(X,\omega)} \leq C(k, X, \omega, \mathrm{Ent})$.

Existence of cscK metrics

The cscK equation for $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ can be written as:

$$F = \log \frac{\omega_\varphi^n}{\omega^n}$$
$$\Delta_\varphi F = -\underline{R} + \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega).$$

Chen-Cheng prove a priori estimates under the assumption of bounded entropy $\mathrm{Ent} = \int_X \log \frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n$:

Chen-Cheng '17

If (X, ω_φ) is a cscK metric, where $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$, then $||\varphi||_{C^k(X,\omega)} \leq C(k, X, \omega, \mathrm{Ent})$.

Existence in Kähler case

Chen-Cheng use their estimates to prove:

Chen-Cheng '18

Properness of Mabuhci energy in terms of L^1 geodesic distance in the space of Kähler potentials $\Rightarrow$ the existence of a cscK metric.

The $\Leftarrow$ direction was established by Berman-Darvas-Lu '16.

Existence in Kähler case

Chen-Cheng use their estimates to prove:

Chen-Cheng '18

Properness of Mabuhci energy in terms of L^1 geodesic distance in the space of Kähler potentials $\Rightarrow$ the existence of a cscK metric.

The $\Leftarrow$ direction was established by Berman-Darvas-Lu '16.

Existence in Kähler case

Chen-Cheng use their estimates to prove:

Chen-Cheng '18

Properness of Mabuhci energy in terms of L^1 geodesic distance in the space of Kähler potentials $\Rightarrow$ the existence of a cscK metric.

The $\Leftarrow$ direction was established by Berman-Darvas-Lu '16.

Obstruction to existence of cscK

Recall that $\underline{R}$ is the average scalar curvature of ω. It follows that

$$\int_X (R - \underline{R})\omega^n = 0 \Rightarrow R - \underline{R} = \Delta_\omega h.$$

Futaki '83

Define the Futaki invariant for the Kähler class $[\omega]$ and a holomorphic vector field V such that $V^i = g^{i\bar{j}}\partial_{\bar{j}} f$, by

$$F(V) = -\int_X V(h)\,\omega^n = \int_X (R - \underline{R}) f \omega^n.$$

If $F(V) \neq 0$ for some V then there cannot exist a cscK metric in $[\omega]$.

Obstruction to existence of cscK

Recall that $\underline{\mathrm{R}}$ is the average scalar curvature of ω. It follows that

$$\int_X (\mathrm{R} - \underline{\mathrm{R}})\omega^n = 0 \Rightarrow \mathrm{R} - \underline{\mathrm{R}} = \Delta_\omega h.$$

Futaki '83

Define the Futaki invariant for the Kähler class $[\omega]$ and a holomorphic vector field V such that $V^i = g^{i\bar{j}}\partial_{\bar{j}} f$, by

$$F(V) = -\int_X V(h)\,\omega^n = \int_X (\mathrm{R} - \underline{\mathrm{R}}) f \omega^n.$$

If $F(V) \neq 0$ for some V then there cannot exist a cscK metric in $[\omega]$.

The Mabuchi energy

Let (X, ω_0) be a compact Kähler manifold. For $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t$, the Mabuchi energy is given in its variational form by

$$\frac{\partial}{\partial t}\,\mathrm{Mab}_{\omega_0}(\omega_t) = -\int_X \dot{\varphi}_t(\mathrm{R}_\varphi - \underline{\mathrm{R}})\omega_t^n,$$

and it can be seen that

$$\mathrm{Mab}_{\omega_0}(\omega_t) = \frac{1}{V}\int_X \log\left(\frac{\omega_t^n}{\omega_0^n}\right)\omega_t^n - \sum_{i=0}^{n-1}\frac{1}{V}\int_X \varphi_t\,\mathrm{Ric}(\omega_0)\wedge\omega_0^i\wedge\omega_t^{n-i-1}$$
$$+\frac{\underline{\mathrm{R}}}{n+1}\sum_{i=0}^{n}\frac{1}{V}\int_X \varphi_t\omega_0^i\wedge\omega_t^{n-i},$$

where $V = \int_X \omega_0^n$.

The Mabuchi energy

Let (X, ω_0) be a compact Kähler manifold. For $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t$, the Mabuchi energy is given in its variational form by

$$\frac{\partial}{\partial t}\mathrm{Mab}_{\omega_0}(\omega_t) = -\int_X \dot{\varphi}_t(\mathrm{R}_\varphi - \underline{\mathrm{R}})\omega_t^n,$$

and it can be seen that

$$\mathrm{Mab}_{\omega_0}(\omega_t) = \frac{1}{V}\int_X \log\left(\frac{\omega_t^n}{\omega_0^n}\right)\omega_t^n - \sum_{i=0}^{n-1}\frac{1}{V}\int_X \varphi_t \,\mathrm{Ric}(\omega_0)\wedge\omega_0^i\wedge\omega_t^{n-i-1}$$
$$+\frac{\underline{\mathrm{R}}}{n+1}\sum_{i=0}^{n}\frac{1}{V}\int_X \varphi_t\omega_0^i\wedge\omega_t^{n-i},$$

where $V = \int_X \omega_0^n$.

The Calabi flow

Let (X, ω_0) be a compact Kähler manifold. The Calabi flow starting at ω_0 is defined by the following evolution

$$\frac{\partial}{\partial t}\omega(t) = \sqrt{-1}\partial\bar{\partial}\,\mathrm{R}(t), \quad \omega(0) = \omega_0.$$

For $\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ with $\varphi \in C^\infty(X)$ normalized so that $\int_X \varphi\omega_\varphi^n = 0$, the Calabi flow can be written as

$$\frac{\partial}{\partial t}\varphi(t) = \mathrm{R}(t) - \underline{\mathrm{R}}, \quad \varphi(0) = 0.$$

- Gradient flow of the Mabuchi energy since

$$\frac{\partial}{\partial t}\mathrm{Mab}_{\omega_0}(\omega(t)) = -\int_X (\mathrm{R}(t) - \underline{\mathrm{R}})^2\omega(t)^n \leq 0$$

- Fixed points are cscK metrics

The Calabi flow

Let (X, ω_0) be a compact Kähler manifold. The Calabi flow starting at ω_0 is defined by the following evolution

$$\frac{\partial}{\partial t}\omega(t) = \sqrt{-1}\partial\bar{\partial}\,\mathrm{R}(t), \quad \omega(0) = \omega_0.$$

For $\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ with $\varphi \in C^\infty(X)$ normalized so that $\int_X \varphi\omega_\varphi^n = 0$, the Calabi flow can be written as

$$\frac{\partial}{\partial t}\varphi(t) = \mathrm{R}(t) - \underline{\mathrm{R}}, \quad \varphi(0) = 0.$$

- Gradient flow of the Mabuchi energy since

$$\frac{\partial}{\partial t}\mathrm{Mab}_{\omega_0}(\omega(t)) = -\int_X (\mathrm{R}(t) - \underline{\mathrm{R}})^2\omega(t)^n \leq 0$$

- Fixed points are cscK metrics

The Calabi flow

Let (X, ω_0) be a compact Kähler manifold. The Calabi flow starting at ω_0 is defined by the following evolution

$$\frac{\partial}{\partial t}\omega(t) = \sqrt{-1}\partial\bar{\partial}\,\mathrm{R}(t), \quad \omega(0) = \omega_0.$$

For $\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ with $\varphi \in C^\infty(X)$ normalized so that $\int_X \varphi\omega_\varphi^n = 0$, the Calabi flow can be written as

$$\frac{\partial}{\partial t}\varphi(t) = \mathrm{R}(t) - \underline{\mathrm{R}}, \quad \varphi(0) = 0.$$

- Gradient flow of the Mabuchi energy since

$$\frac{\partial}{\partial t}\,\mathrm{Mab}_{\omega_0}(\omega(t)) = -\int_X (\mathrm{R}(t) - \underline{\mathrm{R}})^2\omega(t)^n \leq 0$$

- Fixed points are cscK metrics

The Calabi flow

Let (X, ω_0) be a compact Kähler manifold. The Calabi flow starting at ω_0 is defined by the following evolution

$$\frac{\partial}{\partial t}\omega(t) = \sqrt{-1}\partial\bar{\partial}\,\mathrm{R}(t), \quad \omega(0) = \omega_0.$$

For $\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ with $\varphi \in C^\infty(X)$ normalized so that $\int_X \varphi\omega_\varphi^n = 0$, the Calabi flow can be written as

$$\frac{\partial}{\partial t}\varphi(t) = \mathrm{R}(t) - \underline{\mathrm{R}}, \quad \varphi(0) = 0.$$

- Gradient flow of the Mabuchi energy since

$$\frac{\partial}{\partial t}\,\mathrm{Mab}_{\omega_0}(\omega(t)) = -\int_X (\mathrm{R}(t) - \underline{\mathrm{R}})^2 \omega(t)^n \leq 0$$

- Fixed points are cscK metrics

Convergence of the Calabi flow

- Chen-He proved short-time existence of the Calabi flow and global existence of the flow under the assumption of a uniform Ricci bound.
- Székelyhidi proved that uniformly bounded curvature along the flow and proper Mabuchi energy $\Rightarrow$ flow converges to a cscK metric.
- Chen-Sun proved that if initial metric that is sufficiently close to a cscK metric $\Rightarrow$ flow exists and converges uniformly to the cscK metric.

Convergence of the Calabi flow

- Chen-He proved short-time existence of the Calabi flow and global existence of the flow under the assumption of a uniform Ricci bound.
- Székelyhidi proved that uniformly bounded curvature along the flow and proper Mabuchi energy $\Rightarrow$ flow converges to a cscK metric.
- Chen-Sun proved that if initial metric that is sufficiently close to a cscK metric $\Rightarrow$ flow exists and converges uniformly to the cscK metric.

Convergence of the Calabi flow

- Chen-He proved short-time existence of the Calabi flow and global existence of the flow under the assumption of a uniform Ricci bound.
- Székelyhidi proved that uniformly bounded curvature along the flow and proper Mabuchi energy $\Rightarrow$ flow converges to a cscK metric.
- Chen-Sun proved that if initial metric that is sufficiently close to a cscK metric $\Rightarrow$ flow exists and converges uniformly to the cscK metric.

Generalizing to the non-Kähler setting

Question

From this, a natural question would be: to what extent can this theory be extended to non-Kähler Hermitian metrics?

Metrics of constant Chern scalar curvature (CCSC)

- The Chern-Yamabe problem of existence of a CCSC metric within a given Hermitian conformal class was investigated by Angella-Calamai-Spotti '17. They show existence in the case where the expected constant Chern scalar curvature is non-positive.

- Koca-Lejmi '19 explicitly constructed CCSC metrics on ruled surfaces.

Metrics of constant Chern scalar curvature (CCSC)

- The Chern-Yamabe problem of existence of a CCSC metric within a given Hermitian conformal class was investigated by Angella-Calamai-Spotti '17. They show existence in the case where the expected constant Chern scalar curvature is non-positive.
- Koca-Lejmi '19 explicitly constructed CCSC metrics on ruled surfaces.

Metrics of constant Chern scalar curvature (CCSC)

- The Chern-Yamabe problem of existence of a CCSC metric within a given Hermitian conformal class was investigated by Angella-Calamai-Spotti '17. They show existence in the case where the expected constant Chern scalar curvature is non-positive.
- Koca-Lejmi '19 explicitly constructed CCSC metrics on ruled surfaces.

Non-Kähler manifolds with $c_1^{BC}(X) = 0$

The real (1,1) Bott-Chern cohomology is defined by

$$H^{1,1}_{BC}(X, \mathbb{R}) = \frac{\{d\text{-closed real (1,1)-forms}\}}{\{\sqrt{-1}\partial\bar{\partial}\psi,\ \psi \in C^\infty(X, \mathbb{R})\}}.$$

- For non-Kähler manifolds, the condition that $c_1^{BC}(X) = 0$ is a stricter condition than $c_1(X) = 0$. For example, the Hopf surface is diffeomorphic to $S^1 \times S^3$ satisfies $c_1(X) = 0$ but not $c_1^{BC}(X) = 0$.

Cherrier '87, Tosatti-Weinkove '10

For $c_1^{BC}(X) = 0$, there exists a unique Chern Ricci-flat metric in any given $\partial\bar{\partial}$-class of a Hermitian metric.

Non-Kähler manifolds with $c_1^{BC}(X) = 0$

The real (1,1) Bott-Chern cohomology is defined by

$$H^{1,1}_{BC}(X, \mathbb{R}) = \frac{\{d\text{-closed real (1,1)-forms}\}}{\{\sqrt{-1}\partial\bar{\partial}\psi,\ \psi \in C^\infty(X, \mathbb{R})\}}.$$

- For non-Kähler manifolds, the condition that $c_1^{BC}(X) = 0$ is a stricter condition than $c_1(X) = 0$. For example, the Hopf surface is diffeomorphic to $S^1 \times S^3$ satisfies $c_1(X) = 0$ but not $c_1^{BC}(X) = 0$.

Cherrier '87, Tosatti-Weinkove '10

For $c_1^{BC}(X) = 0$, there exists a unique Chern Ricci-flat metric in any given $\partial\bar{\partial}$-class of a Hermitian metric.

Mabuchi energy on Hermitian manifolds

Let (X, ω_0) compact, Hermitian manifold with $c_1^{BC}(X) = 0$ ($\underline{R} = 0$). Tosatti-Weinkove observed that Mabuchi energy can be defined by

$$\mathrm{Mab}_{\omega_0}(\omega_\varphi) = \frac{1}{V}\int_X \left(\log \frac{\omega_\varphi^n}{\omega_0^n} - F\right)\omega_\varphi^n + \frac{1}{V}\int_X F\omega_0^n,$$

where $V = \int_X \omega_0^n$ and F is the Chern-Ricci potential of ω_0

$$\mathrm{Ric}(\omega_0) = \sqrt{-1}\partial\bar{\partial}F, \quad \int_X e^F \omega_0^n = \int_X \omega_0^n.$$

- If ω_0 is Kähler, then this agrees with the standard Mabuchi energy in the Kähler setting.
- Critical points are Chern-Ricci-flat metrics.

Mabuchi energy on Hermitian manifolds

Let (X, ω_0) compact, Hermitian manifold with $c_1^{\mathrm{BC}}(X) = 0$ ($\underline{\mathrm{R}} = 0$). Tosatti-Weinkove observed that Mabuchi energy can be defined by

$$\mathrm{Mab}_{\omega_0}(\omega_\varphi) = \frac{1}{V}\int_X \left(\log \frac{\omega_\varphi^n}{\omega_0^n} - F\right)\omega_\varphi^n + \frac{1}{V}\int_X F\omega_0^n,$$

where $V = \int_X \omega_0^n$ and F is the Chern-Ricci potential of ω_0

$$\mathrm{Ric}(\omega_0) = \sqrt{-1}\partial\bar{\partial}F, \quad \int_X e^F \omega_0^n = \int_X \omega_0^n.$$

- If ω_0 is Kähler, then this agrees with the standard Mabuchi energy in the Kähler setting.
- Critical points are Chern-Ricci-flat metrics.

Mabuchi energy on Hermitian manifolds

Let (X, ω_0) compact, Hermitian manifold with $c_1^{\mathrm{BC}}(X) = 0$ ($\underline{\mathrm{R}} = 0$). Tosatti-Weinkove observed that Mabuchi energy can be defined by

$$\mathrm{Mab}_{\omega_0}(\omega_\varphi) = \frac{1}{V}\int_X \left(\log\frac{\omega_\varphi^n}{\omega_0^n} - F\right)\omega_\varphi^n + \frac{1}{V}\int_X F\omega_0^n,$$

where $V = \int_X \omega_0^n$ and F is the Chern-Ricci potential of ω_0

$$\mathrm{Ric}(\omega_0) = \sqrt{-1}\partial\bar{\partial}F, \quad \int_X e^F\omega_0^n = \int_X \omega_0^n.$$

- If ω_0 is Kähler, then this agrees with the standard Mabuchi energy in the Kähler setting.
- Critical points are Chern-Ricci-flat metrics.

Mabuchi energy on Hermitian manifolds

Let (X, ω_0) compact, Hermitian manifold with $c_1^{\mathrm{BC}}(X) = 0$ ($\underline{\mathrm{R}} = 0$). Tosatti-Weinkove observed that Mabuchi energy can be defined by

$$\mathrm{Mab}_{\omega_0}(\omega_\varphi) = \frac{1}{V}\int_X \left(\log \frac{\omega_\varphi^n}{\omega_0^n} - F\right)\omega_\varphi^n + \frac{1}{V}\int_X F\omega_0^n,$$

where $V = \int_X \omega_0^n$ and F is the Chern-Ricci potential of ω_0

$$\mathrm{Ric}(\omega_0) = \sqrt{-1}\partial\bar{\partial}F, \quad \int_X e^F \omega_0^n = \int_X \omega_0^n.$$

- If ω_0 is Kähler, then this agrees with the standard Mabuchi energy in the Kähler setting.
- Critical points are Chern-Ricci-flat metrics.

Mabuchi energy on Hermitian manifolds cont.

When $c_1^{\mathrm{BC}}(X) = 0$,

$$\frac{\partial}{\partial t} \mathrm{Mab}_{\omega_0}(\omega_\varphi) = \int_X \log\left(\frac{\omega_\varphi^n}{e^F \omega_0^n}\right) \Delta_\varphi \dot{\varphi} \omega_\varphi^n.$$

1. Taking $\dot{\varphi} = \log\left(\frac{\omega_\varphi^n}{e^F \omega_0^n}\right)$ gives you the Chern-Ricci flow and we have that

$$\frac{\partial}{\partial t} \mathrm{Mab}_{\omega_0}(\omega_\varphi) = -\int_X |\nabla \log\left(\frac{\omega_\varphi^n}{e^F \omega_0^n}\right)|_\varphi^2 \omega_\varphi^n \leq 0.$$

2. Taking $\dot{\varphi} = -\Delta_\varphi \log\left(\frac{\omega_\varphi^n}{e^F \omega_0^n}\right) + \textit{torsion} = -\mathrm{R}_\varphi + \textit{torsion}$ gives you a Chern-Calabi flow which satisfies

$$\frac{\partial}{\partial t} \mathrm{Mab}_{\omega_0}(\omega_\varphi) = \int_X -(\mathrm{R}_\varphi + \textit{torsion})^2 \omega_\varphi^n \leq 0.$$

Mabuchi energy on Hermitian manifolds cont.

When $c_1^{\mathrm{BC}}(X) = 0$,

$$\frac{\partial}{\partial t}\,\mathrm{Mab}_{\omega_0}(\omega_\varphi) = \int_X \log\left(\frac{\omega_\varphi^n}{e^F\omega_0^n}\right)\Delta_\varphi\dot{\varphi}\omega_\varphi^n.$$

1. Taking $\dot{\varphi} = \log\left(\frac{\omega_\varphi^n}{e^F\omega_0^n}\right)$ gives you the Chern-Ricci flow and we have that

$$\frac{\partial}{\partial t}\,\mathrm{Mab}_{\omega_0}(\omega_\varphi) = -\int_X |\nabla\log\left(\frac{\omega_\varphi^n}{e^F\omega_0^n}\right)|_\varphi^2\omega_\varphi^n \leq 0.$$

2. Taking $\dot{\varphi} = -\Delta_\varphi \log\left(\frac{\omega_\varphi^n}{e^F\omega_0^n}\right) + \mathit{torsion} = -\mathrm{R}_\varphi + \mathit{torsion}$ gives you a Chern-Calabi flow which satisfies

$$\frac{\partial}{\partial t}\,\mathrm{Mab}_{\omega_0}(\omega_\varphi) = \int_X -(\mathrm{R}_\varphi + \mathit{torsion})^2\omega_\varphi^n \leq 0.$$

Mabuchi energy on Hermitian manifolds cont.

When $c_1^{\mathrm{BC}}(X) = 0$,

$$\frac{\partial}{\partial t}\operatorname{Mab}_{\omega_0}(\omega_\varphi) = \int_X \log\left(\frac{\omega_\varphi^n}{e^F\omega_0^n}\right)\Delta_\varphi\dot{\varphi}\omega_\varphi^n.$$

1. Taking $\dot{\varphi} = \log\left(\frac{\omega_\varphi^n}{e^F\omega_0^n}\right)$ gives you the Chern-Ricci flow and we have that

$$\frac{\partial}{\partial t}\operatorname{Mab}_{\omega_0}(\omega_\varphi) = -\int_X |\nabla\log\left(\frac{\omega_\varphi^n}{e^F\omega_0^n}\right)|_\varphi^2\omega_\varphi^n \leq 0.$$

2. Taking $\dot{\varphi} = -\Delta_\varphi\log\left(\frac{\omega_\varphi^n}{e^F\omega_0^n}\right) + \textit{torsion} = -\mathrm{R}_\varphi + \textit{torsion}$ gives you a Chern-Calabi flow which satisfies

$$\frac{\partial}{\partial t}\operatorname{Mab}_{\omega_0}(\omega_\varphi) = \int_X -(\mathrm{R}_\varphi + \textit{torsion})^2\omega_\varphi^n \leq 0.$$

Generalizing to the non-Kähler setting

- We need to make the assumption that $\partial\bar{\partial}\omega_0^k = 0$ for $k = 1, 2$ to ensure:
  - Volume is preserved: $\int_X (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \int_X \omega^n$
  - Average scalar curvature is invariant: $\underline{R(\omega)} = \frac{\int_X R(\omega)\omega^n}{\int_X \omega^n} = \frac{\int_X R(\omega_\varphi)\omega_\varphi^n}{\int_X \omega^n} = \underline{R(\omega_\varphi)}$
  - Integrals of Laplacians vanish: $\int_X \Delta_\omega f \omega^n = 0$
- For complex surfaces, this condition coincides with the Gauduchon condition $\partial\bar{\partial}\omega^{n-1} = 0$.

Generalizing to the non-Kähler setting

- We need to make the assumption that $\partial\bar{\partial}\omega_0^k = 0$ for $k = 1, 2$ to ensure:
  - Volume is preserved: $\int_X (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \int_X \omega^n$
  - Average scalar curvature is invariant: $\underline{R(\omega)} = \frac{\int_X R(\omega)\omega^n}{\int_X \omega^n} = \frac{\int_X R(\omega_\varphi)\omega_\varphi^n}{\int_X \omega^n} = \underline{R(\omega_\varphi)}$
  - Integrals of Laplacians vanish: $\int_X \Delta_\omega f \omega^n = 0$
- For complex surfaces, this condition coincides with the Gauduchon condition $\partial\bar{\partial}\omega^{n-1} = 0$.

Generalizing to the non-Kähler setting

- We need to make the assumption that $\partial\bar{\partial}\omega_0^k = 0$ for $k = 1, 2$ to ensure:
  - Volume is preserved: $\int_X (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \int_X \omega^n$
  - Average scalar curvature is invariant: $\underline{\mathrm{R}(\omega)} = \frac{\int_X \mathrm{R}(\omega)\omega^n}{\int_X \omega^n} = \frac{\int_X \mathrm{R}(\omega_\varphi)\omega_\varphi^n}{\int_X \omega^n} = \underline{\mathrm{R}(\omega_\varphi)}$
  - Integrals of Laplacians vanish: $\int_X \Delta_\omega f \omega^n = 0$
- For complex surfaces, this condition coincides with the Gauduchon condition $\partial\bar{\partial}\omega^{n-1} = 0$.

Generalizing to the non-Kähler setting

- We need to make the assumption that $\partial\bar{\partial}\omega_0^k = 0$ for $k = 1, 2$ to ensure:
  - Volume is preserved: $\int_X (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \int_X \omega^n$
  - Average scalar curvature is invariant: $\underline{\mathrm{R}(\omega)} = \frac{\int_X \mathrm{R}(\omega)\omega^n}{\int_X \omega^n} = \frac{\int_X \mathrm{R}(\omega_\varphi)\omega_\varphi^n}{\int_X \omega^n} = \underline{\mathrm{R}(\omega_\varphi)}$
  - Integrals of Laplacians vanish: $\int_X \Delta_\omega f \omega^n = 0$
- For complex surfaces, this condition coincides with the Gauduchon condition $\partial\bar{\partial}\omega^{n-1} = 0$.

Generalizing to the non-Kähler setting

- We need to make the assumption that $\partial\bar{\partial}\omega_0^k = 0$ for $k = 1, 2$ to ensure:
  - Volume is preserved: $\int_X (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \int_X \omega^n$
  - Average scalar curvature is invariant: $\underline{\mathrm{R}(\omega)} = \frac{\int_X \mathrm{R}(\omega)\omega^n}{\int_X \omega^n} = \frac{\int_X \mathrm{R}(\omega_\varphi)\omega_\varphi^n}{\int_X \omega^n} = \underline{\mathrm{R}(\omega_\varphi)}$
  - Integrals of Laplacians vanish: $\int_X \Delta_\omega f \omega^n = 0$
- For complex surfaces, this condition coincides with the Gauduchon condition $\partial\bar{\partial}\omega^{n-1} = 0$.

Assumption on ω

The assumption that $\partial\bar{\partial}\omega^k = 0$ for $k = 1, 2$ in fact implies that $\partial\bar{\partial}\omega^k = 0$ for $k = 1, \ldots, n-1$ since

$$\partial\bar{\partial}\omega^k = k\partial\bar{\partial}\omega \wedge \omega^{k-1} - k(k-1)\bar{\partial}\omega \wedge \partial\omega \wedge \omega^{k-2}$$

and the *blue* terms vanish since

$$0 = \partial\bar{\partial}\omega^2 = 2(\partial\bar{\partial}\omega \wedge \omega + \partial\omega \wedge \bar{\partial}\omega) = 2\partial\omega \wedge \bar{\partial}\omega$$

and $\partial\bar{\partial}\omega = 0$ by assumption.

Assumption on ω

The assumption that $\partial\bar{\partial}\omega^k = 0$ for $k = 1, 2$ in fact implies that $\partial\bar{\partial}\omega^k = 0$ for $k = 1, \ldots, n-1$ since

$$\partial\bar{\partial}\omega^k = k\partial\bar{\partial}\omega \wedge \omega^{k-1} - k(k-1)\bar{\partial}\omega \wedge \partial\omega \wedge \omega^{k-2}$$

and the *blue* terms vanish since

$$0 = \partial\bar{\partial}\omega^2 = 2(\partial\bar{\partial}\omega \wedge \omega + \partial\omega \wedge \bar{\partial}\omega) = 2\partial\omega \wedge \bar{\partial}\omega$$

and $\partial\bar{\partial}\omega = 0$ by assumption.

Assumption on ω

The assumption that $\partial\bar{\partial}\omega^k = 0$ for $k = 1, 2$ in fact implies that $\partial\bar{\partial}\omega^k = 0$ for $k = 1, \ldots, n-1$ since

$$\partial\bar{\partial}\omega^k = k\partial\bar{\partial}\omega \wedge \omega^{k-1} - k(k-1)\bar{\partial}\omega \wedge \partial\omega \wedge \omega^{k-2}$$

and the *blue* terms vanish since

$$0 = \partial\bar{\partial}\omega^2 = 2(\partial\bar{\partial}\omega \wedge \omega + \partial\omega \wedge \bar{\partial}\omega) = 2\partial\omega \wedge \bar{\partial}\omega$$

and $\partial\bar{\partial}\omega = 0$ by assumption.

Assumption on ω cont.

Theorem (Gauduchon '77)

Every Hermitian metric is conformal to a Gauduchon metric.

Example: Let $X = N \times M$ for N a complex surface and M Kähler. Then taking ω_N to be a Gauduchon metric on N which always exists by Gauduchon's theorem and ω_M Kähler, the metric $\omega = \omega_N + \omega_M$ satisfies $\partial\bar{\partial}\omega^k = 0$ for $k = 1, 2$.

Assumption on ω cont.

Theorem (Gauduchon '77)

Every Hermitian metric is conformal to a Gauduchon metric.

Example: Let $X = N \times M$ for N a complex surface and M Kähler. Then taking ω_N to be a Gauduchon metric on N which always exists by Gauduchon's theorem and ω_M Kähler, the metric $\omega = \omega_N + \omega_M$ satisfies $\partial\bar{\partial}\omega^k = 0$ for $k = 1, 2$.

Estimates for CCSC metrics

We have the following a priori estimates:

Theorem (S - '19)

Let (X, ω) be a compact, complex manifold with ω a Hermitian metric satisfying $\partial\bar{\partial}\omega = \partial\bar{\partial}\omega^2 = 0$. If $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a metric of constant Chern scalar curvature then for $k = 0, 1, 2, \ldots$, we have

$$||\varphi||_{C^k(X,\omega)} \leq C(k, X, \omega, \mathrm{Ent}),$$

where $\mathrm{Ent} := \int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\omega_\varphi^n$ is the entropy of ω, ω_φ.

Proposition (S - '20)

If $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for a smooth potential function φ on X, then for all k, we have $||\varphi||_{C^k(X,\omega)} \leq C(k, X, \omega, \mathrm{Ent}, ||\,\mathrm{R}_\varphi\,||_0)$.

Estimates for CCSC metrics

We have the following a priori estimates:

Theorem (S - '19)

Let (X, ω) be a compact, complex manifold with ω a Hermitian metric satisfying $\partial\bar{\partial}\omega = \partial\bar{\partial}\omega^2 = 0$. If $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a metric of constant Chern scalar curvature then for $k = 0, 1, 2, \ldots$, we have

$$||\varphi||_{C^k(X,\omega)} \leq C(k, X, \omega, \text{Ent}),$$

where $\text{Ent} := \int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\omega_\varphi^n$ is the entropy of ω, ω_φ.

Proposition (S - '20)

If $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for a smooth potential function φ on X, then for all k, we have $||\varphi||_{C^k(X,\omega)} \leq C(k, X, \omega, \text{Ent}, ||\,\mathrm{R}_\varphi\,||_0)$.

Estimates for CCSC metrics

We have the following a priori estimates:

Theorem (S - '19)

Let (X, ω) be a compact, complex manifold with ω a Hermitian metric satisfying $\partial\bar{\partial}\omega = \partial\bar{\partial}\omega^2 = 0$. If $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a metric of constant Chern scalar curvature then for $k = 0, 1, 2, \ldots$, we have

$$||\varphi||_{C^k(X,\omega)} \leq C(k, X, \omega, \mathrm{Ent}),$$

where $\mathrm{Ent} := \int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\omega_\varphi^n$ is the entropy of ω, ω_φ.

Proposition (S - '20)

If $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for a smooth potential function φ on X, then for all k, we have $||\varphi||_{C^k(X,\omega)} \leq C(k, X, \omega, \mathrm{Ent}, ||\,\mathrm{R}_\varphi\,||_0)$.

A Chern-Calabi flow on Hermitian manifolds

We introduce a **Chern-Calabi flow** of $\omega_\varphi(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ starting at ω_0 evolving by

$$\frac{\partial}{\partial t}\varphi(t) = \mathrm{R}(t) + 2\,\mathrm{Re}\langle(\mathrm{tr}\,T(t)), \partial \log \frac{\omega_\varphi^n}{e^F \omega_0^n}\rangle_{\omega(t)},$$

where where F is the Chern-Ricci potential of ω_0 and $(\mathrm{tr}\,T)_* = T^i_{i*}$ is the trace of the torsion of $\omega(t)$.

- If ω_0 is Kähler then this agrees with the Calabi flow.
- Gradient flow of the Mabuchi energy for Hermitian manifolds with $c_1^{BC}(X) = 0$.
- Fixed points are constant Chern scalar curvature (CCSC) metrics which in this case are Chern-Ricci-flat.

A Chern-Calabi flow on Hermitian manifolds

We introduce a **Chern-Calabi flow** of $\omega_\varphi(t) = \omega_0 + \sqrt{-1}\partial\bar\partial\varphi(t)$ starting at ω_0 evolving by

$$\frac{\partial}{\partial t}\varphi(t) = \mathrm{R}(t) + 2\,\mathrm{Re}\langle(\mathrm{tr}\,T(t)), \partial \log \frac{\omega_\varphi^n}{e^F\omega_0^n}\rangle_{\omega(t)},$$

where where F is the Chern-Ricci potential of ω_0 and $(\mathrm{tr}\,T)_* = T^i_{i*}$ is the trace of the torsion of $\omega(t)$.

- If ω_0 is Kähler then this agrees with the Calabi flow.
- Gradient flow of the Mabuchi energy for Hermitian manifolds with $c_1^{BC}(X) = 0$.
- Fixed points are constant Chern scalar curvature (CCSC) metrics which in this case are Chern-Ricci-flat.

A Chern-Calabi flow on Hermitian manifolds

We introduce a **Chern-Calabi flow** of $\omega_\varphi(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ starting at ω_0 evolving by

$$\frac{\partial}{\partial t}\varphi(t) = \mathrm{R}(t) + 2\,\mathrm{Re}\langle(\mathrm{tr}\,T(t)), \partial \log \frac{\omega_\varphi^n}{e^F\omega_0^n}\rangle_{\omega(t)},$$

where where F is the Chern-Ricci potential of ω_0 and $(\mathrm{tr}\,T)_* = T^i_{i*}$ is the trace of the torsion of $\omega(t)$.

- If ω_0 is Kähler then this agrees with the Calabi flow.
- Gradient flow of the Mabuchi energy for Hermitian manifolds with $c_1^{BC}(X) = 0$.
- Fixed points are constant Chern scalar curvature (CCSC) metrics which in this case are Chern-Ricci-flat.

A Chern-Calabi flow on Hermitian manifolds

We introduce a **Chern-Calabi flow** of $\omega_\varphi(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ starting at ω_0 evolving by

$$\frac{\partial}{\partial t}\varphi(t) = \mathrm{R}(t) + 2\,\mathrm{Re}\langle(\mathrm{tr}\,T(t)), \partial \log \frac{\omega_\varphi^n}{e^F \omega_0^n}\rangle_{\omega(t)},$$

where where F is the Chern-Ricci potential of ω_0 and $(\mathrm{tr}\,T)_* = T^i_{i*}$ is the trace of the torsion of $\omega(t)$.

- If ω_0 is Kähler then this agrees with the Calabi flow.
- Gradient flow of the Mabuchi energy for Hermitian manifolds with $c_1^{\mathrm{BC}}(X) = 0$.
- Fixed points are constant Chern scalar curvature (CCSC) metrics which in this case are Chern-Ricci-flat.

Convergence of Chern-Calabi flow

Let (X, ω_0) be a compact Hermitian manifold with $c_1^{BC}(X) = 0$ and $\partial\bar{\partial}\omega_0 = \partial\bar{\partial}\omega_0^2 = 0$.

Theorem (S - '20)

A solution to the Chern-Calabi flow starting at ω_0 exists as long as the Chern scalar curvature remains bounded. If the Chern scalar curvature remains uniformly bounded for all time, we have smooth convergence of the flow to the unique Chern-Ricci-flat metric of the form $\omega_\infty = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\infty$.

$$\begin{aligned}\mathrm{Ent}(\omega, \omega_\varphi) := \int_X \log \frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n &= \mathrm{Mab}_\omega(\omega_\varphi) - \int_X F\omega_\varphi^n + \int_X F\omega^n \\ &\leq \mathrm{Mab}_\omega(\omega_\varphi) + C.\end{aligned}$$

Convergence of Chern-Calabi flow

Let (X, ω_0) be a compact Hermitian manifold with $c_1^{BC}(X) = 0$ and $\partial\bar{\partial}\omega_0 = \partial\bar{\partial}\omega_0^2 = 0$.

Theorem (S - '20)

A solution to the Chern-Calabi flow starting at ω_0 exists as long as the Chern scalar curvature remains bounded. If the Chern scalar curvature remains uniformly bounded for all time, we have smooth convergence of the flow to the unique Chern-Ricci-flat metric of the form $\omega_\infty = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\infty$.

$$\mathrm{Ent}(\omega, \omega_\varphi) := \int_X \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n = \mathrm{Mab}_\omega(\omega_\varphi) - \int_X F\omega_\varphi^n + \int_X F\omega^n$$
$$\leq \mathrm{Mab}_\omega(\omega_\varphi) + C.$$

Convergence of Chern-Calabi flow

Let (X, ω_0) be a compact Hermitian manifold with $c_1^{BC}(X) = 0$ and $\partial\bar{\partial}\omega_0 = \partial\bar{\partial}\omega_0^2 = 0$.

Theorem (S - '20)

A solution to the Chern-Calabi flow starting at ω_0 exists as long as the Chern scalar curvature remains bounded. If the Chern scalar curvature remains uniformly bounded for all time, we have smooth convergence of the flow to the unique Chern-Ricci-flat metric of the form $\omega_\infty = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\infty$.

$$\begin{aligned}\mathrm{Ent}(\omega, \omega_\varphi) := \int_X \log \frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n &= \mathrm{Mab}_\omega(\omega_\varphi) - \int_X F\omega_\varphi^n + \int_X F\omega^n \\ &\leq \mathrm{Mab}_\omega(\omega_\varphi) + C.\end{aligned}$$

The flow preserves the $\partial\bar{\partial}\omega^k = 0$ property

The condition that $\partial\bar{\partial}\omega_\varphi = 0$ is also preserved by the flow since

$$\frac{d}{dt}\partial\bar{\partial}\omega_\varphi = \partial\bar{\partial}\big(\sqrt{-1}\partial\bar{\partial}(\mathrm{R}_\varphi + 2\,\mathrm{Re}\langle \mathrm{tr}\, T_\varphi, \partial(\log \frac{\omega_\varphi^n}{e^F\omega_0^n})\rangle_\varphi)\big) = 0.$$

Similarly $\partial\bar{\partial}\omega_\varphi^2 = 0$ is preserved since

$$\frac{d}{dt}\partial\bar{\partial}\omega_\varphi^2 = 2\partial\bar{\partial}\omega_\varphi \wedge \frac{d}{dt}\partial\bar{\partial}\omega_\varphi = 0.$$

Together, this gives us that $\partial\bar{\partial}\omega_\varphi^k = 0$ along the flow if $\partial\bar{\partial}\omega_0^k = 0$.

The flow preserves the $\partial\bar{\partial}\omega^k = 0$ property

The condition that $\partial\bar{\partial}\omega_\varphi = 0$ is also preserved by the flow since

$$\frac{d}{dt}\partial\bar{\partial}\omega_\varphi = \partial\bar{\partial}\big(\sqrt{-1}\partial\bar{\partial}(\mathrm{R}_\varphi + 2\,\mathrm{Re}\langle \mathrm{tr}\, T_\varphi, \partial(\log \frac{\omega_\varphi^n}{e^F\omega_0^n})\rangle_\varphi)\big) = 0.$$

Similarly $\partial\bar{\partial}\omega_\varphi^2 = 0$ is preserved since

$$\frac{d}{dt}\partial\bar{\partial}\omega_\varphi^2 = 2\partial\bar{\partial}\omega_\varphi \wedge \frac{d}{dt}\partial\bar{\partial}\omega_\varphi = 0.$$

Together, this gives us that $\partial\bar{\partial}\omega_\varphi^k = 0$ along the flow if $\partial\bar{\partial}\omega_0^k = 0$.

The flow preserves the $\partial\bar{\partial}\omega^k = 0$ property

The condition that $\partial\bar{\partial}\omega_\varphi = 0$ is also preserved by the flow since

$$\frac{d}{dt}\partial\bar{\partial}\omega_\varphi = \partial\bar{\partial}\big(\sqrt{-1}\partial\bar{\partial}(\mathrm{R}_\varphi + 2\,\mathrm{Re}\langle \mathrm{tr}\, T_\varphi, \partial(\log \frac{\omega_\varphi^n}{e^F\omega_0^n})\rangle_\varphi)\big) = 0.$$

Similarly $\partial\bar{\partial}\omega_\varphi^2 = 0$ is preserved since

$$\frac{d}{dt}\partial\bar{\partial}\omega_\varphi^2 = 2\partial\bar{\partial}\omega_\varphi \wedge \frac{d}{dt}\partial\bar{\partial}\omega_\varphi = 0.$$

Together, this gives us that $\partial\bar{\partial}\omega_\varphi^k = 0$ along the flow if $\partial\bar{\partial}\omega_0^k = 0$.

Outline of the proof of the estimates

The fourth order CCSC equation can be written as a pair of coupled second order equations:

$$F = \log \frac{\omega_\varphi^n}{\omega^n}$$
$$\Delta_\varphi F = -\underline{\mathrm{R}} + \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega)$$

where $\Delta_\varphi, \mathrm{Ric}(\omega)$ denote the Chern Laplacian and Chern-Ricci curvature, respectively. Following a similar approach to that of Chen-Cheng we prove:

1. $|F| + |\varphi| \leq C$
2. $|\partial\varphi|^2_\omega \leq C$
3. $||\,\mathrm{tr}_\omega \omega_\varphi||_{L^p(X,\omega)} \leq C(p)$
4. $\mathrm{tr}_\omega \omega_\varphi \leq C$

where C depends on X, ω and the entropy.

Outline of the proof of the estimates

The fourth order CCSC equation can be written as a pair of coupled second order equations:

$$F = \log \frac{\omega_\varphi^n}{\omega^n}$$
$$\Delta_\varphi F = -\underline{R} + \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega)$$

where $\Delta_\varphi, \mathrm{Ric}(\omega)$ denote the Chern Laplacian and Chern-Ricci curvature, respectively. Following a similar approach to that of Chen-Cheng we prove:

1. $|F| + |\varphi| \leq C$
2. $|\partial\varphi|^2_\omega \leq C$
3. $||\,\mathrm{tr}_\omega \omega_\varphi||_{L^p(X,\omega)} \leq C(p)$
4. $\mathrm{tr}_\omega \omega_\varphi \leq C$

where C depends on X, ω and the entropy.

Outline of the proof of the estimates

The fourth order CCSC equation can be written as a pair of coupled second order equations:

$$F = \log \frac{\omega_\varphi^n}{\omega^n}$$

$$\Delta_\varphi F = -\underline{\mathrm{R}} + \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega)$$

where $\Delta_\varphi, \mathrm{Ric}(\omega)$ denote the Chern Laplacian and Chern-Ricci curvature, respectively. Following a similar approach to that of Chen-Cheng we prove:

1. $|F| + |\varphi| \leq C$
2. $|\partial\varphi|^2_\omega \leq C$
3. $|| \mathrm{tr}_\omega \omega_\varphi ||_{L^p(X,\omega)} \leq C(p)$
4. $\mathrm{tr}_\omega \omega_\varphi \leq C$

where C depends on X, ω and the entropy.

Outline of the proof of the estimates

The fourth order CCSC equation can be written as a pair of coupled second order equations:

$$F = \log \frac{\omega_\varphi^n}{\omega^n}$$
$$\Delta_\varphi F = -\underline{\mathrm{R}} + \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega)$$

where $\Delta_\varphi, \mathrm{Ric}(\omega)$ denote the Chern Laplacian and Chern-Ricci curvature, respectively. Following a similar approach to that of Chen-Cheng we prove:

1. $|F| + |\varphi| \leq C$
2. $|\partial\varphi|^2_\omega \leq C$
3. $|| \mathrm{tr}_\omega \omega_\varphi ||_{L^p(X,\omega)} \leq C(p)$
4. $\mathrm{tr}_\omega \omega_\varphi \leq C$

where C depends on X, ω and the entropy.

Outline of the proof of the estimates

The fourth order CCSC equation can be written as a pair of coupled second order equations:

$$F = \log \frac{\omega_\varphi^n}{\omega^n}$$

$$\Delta_\varphi F = -\underline{\mathrm{R}} + \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega)$$

where $\Delta_\varphi, \mathrm{Ric}(\omega)$ denote the Chern Laplacian and Chern-Ricci curvature, respectively. Following a similar approach to that of Chen-Cheng we prove:

1. $|F| + |\varphi| \leq C$
2. $|\partial\varphi|^2_\omega \leq C$
3. $|| \mathrm{tr}_\omega \omega_\varphi ||_{L^p(X,\omega)} \leq C(p)$
4. $\mathrm{tr}_\omega \omega_\varphi \leq C$

where C depends on X, ω and the entropy.

Outline of the proof of the estimates

The fourth order CCSC equation can be written as a pair of coupled second order equations:

$$F = \log \frac{\omega_\varphi^n}{\omega^n}$$

$$\Delta_\varphi F = -\underline{\mathrm{R}} + \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega)$$

where $\Delta_\varphi, \mathrm{Ric}(\omega)$ denote the Chern Laplacian and Chern-Ricci curvature, respectively. Following a similar approach to that of Chen-Cheng we prove:

1. $|F| + |\varphi| \leq C$
2. $|\partial\varphi|^2_\omega \leq C$
3. $||\,\mathrm{tr}_\omega \omega_\varphi||_{L^p(X,\omega)} \leq C(p)$
4. $\mathrm{tr}_\omega \omega_\varphi \leq C$

where C depends on X, ω and the entropy.

Bootstrapping

From the uniform bound on $\mathrm{tr}_\omega\, \omega_\varphi$, we can in fact show that

$$\mathrm{tr}_{\omega_\varphi}\, \omega \leq \frac{1}{(n-1)!}(\mathrm{tr}_\omega\, \omega_\varphi)^{n-1}\frac{\omega^n}{\omega_\varphi^n} \leq C$$

A bootstrapping argument using the coupled second order equations gives $||\varphi||^2_{C^k(X,\omega)}$ for all $k \in \mathbb{N}$ by applying standard elliptic PDE theory.

Bootstrapping

From the uniform bound on $\operatorname{tr}_\omega \omega_\varphi$, we can in fact show that

$$\operatorname{tr}_{\omega_\varphi} \omega \leq \frac{1}{(n-1)!}(\operatorname{tr}_\omega \omega_\varphi)^{n-1} \frac{\omega^n}{\omega_\varphi^n} \leq C$$

A bootstrapping argument using the coupled second order equations gives $||\varphi||^2_{C^k(X,\omega)}$ for all $k \in \mathbb{N}$ by applying standard elliptic PDE theory.

Notable differences in the non-Kähler setting

- Torsion terms that arise from commuting covariant derivatives $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$ no longer vanish.
- Additional term from integration by parts:

$$
\begin{aligned}
0 &= \int_X \partial(f \bar{\partial} g \wedge \omega^{n-1}) \\
&= \int_X \partial f \wedge \bar{\partial} g \wedge \omega^{n-1} + \int_X f \partial\bar{\partial} g \wedge \omega^{n-1} - \int_X f \bar{\partial} g \wedge \partial(\omega^{n-1})
\end{aligned}
$$

Notable differences in the non-Kähler setting

- Torsion terms that arise from commuting covariant derivatives $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$ no longer vanish.
- Additional term from integration by parts:

$$0 = \int_X \partial(f\bar{\partial}g \wedge \omega^{n-1})$$

$$= \int_X \partial f \wedge \bar{\partial}g \wedge \omega^{n-1} + \int_X f\partial\bar{\partial}g \wedge \omega^{n-1} - \int_X f\bar{\partial}g \wedge \partial(\omega^{n-1})$$

Notable differences in the non-Kähler setting

- Torsion terms that arise from commuting covariant derivatives $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$ no longer vanish.
- Additional term from integration by parts:

$$
\begin{aligned}
0 &= \int_X \partial(f\bar{\partial}g \wedge \omega^{n-1}) \\
&= \int_X \partial f \wedge \bar{\partial}g \wedge \omega^{n-1} + \int_X f\partial\bar{\partial}g \wedge \omega^{n-1} - \int_X f\bar{\partial}g \wedge \partial(\omega^{n-1})
\end{aligned}
$$

Step 1: C^0 estimates on F and φ

Recall the CCSC equations:

$$F = \log \frac{\omega_\varphi^n}{\omega^n}$$
$$\Delta_\varphi F = -\underline{R} + \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega).$$

Lemma

Let (φ, F) be a smooth solution to CCSC, then there exists a C depending only on (X, ω) and an upper bound on the entropy such that $|F| + |\varphi| \leq C$ for all x on X.

Proof: Firstly, let us normalize φ so that $\sup_X \varphi = 0$ and ω such that $\int_X \omega^n = 1$.

Step 1: C^0 estimates on F and φ

Recall the CCSC equations:

$$F = \log \frac{\omega_\varphi^n}{\omega^n}$$
$$\Delta_\varphi F = -\underline{R} + \operatorname{tr}_{\omega_\varphi} \operatorname{Ric}(\omega).$$

Lemma

Let (φ, F) be a smooth solution to CCSC, then there exists a C depending only on (X, ω) and an upper bound on the entropy such that $|F| + |\varphi| \leq C$ for all x on X.

Proof: Firstly, let us normalize φ so that $\sup_X \varphi = 0$ and ω such that $\int_X \omega^n = 1$.

Step 1: C^0 estimates on F and φ

Recall the CCSC equations:

$$F = \log \frac{\omega_\varphi^n}{\omega^n}$$
$$\Delta_\varphi F = -\underline{R} + \mathrm{tr}_{\omega_\varphi} \mathrm{Ric}(\omega).$$

Lemma

Let (φ, F) be a smooth solution to CCSC, then there exists a C depending only on (X, ω) and an upper bound on the entropy such that $|F| + |\varphi| \leq C$ for all x on X.

Proof: Firstly, let us normalize φ so that $\sup_X \varphi = 0$ and ω such that $\int_X \omega^n = 1$.

Yau's theorem for complex manifolds

Tosatti-Weinkove '10

For every smooth real-valued function G on X there exist a unique $b \in \mathbb{R}$ and a unique smooth real-valued function ψ on X solving

$$(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{G+b}\omega^n,$$
$$\text{with } \omega + \sqrt{-1}\partial\bar{\partial}\psi > 0, \quad \sup_X \psi = 0.$$

In particular, when $\partial\bar{\partial}\omega^k = 0$, for $k = 1, 2$, then the constant b must equal

$$\log \frac{\int_X \omega^n}{\int_X e^G \omega^n} .$$

Using the previous theorem with $G = F\log\sqrt{F^2+1}$, let ψ be the unique function with $\sup_X \psi = 0$ and ψ solving

$$(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{G+b}\omega^n = \frac{e^F\sqrt{F^2+1}\omega^n}{\int_X e^F\sqrt{F^2+1}\omega^n}.$$

Tian's α-invariant

Given (X, ω) a complex manifold, there exist constants $\alpha > 0$ and $C > 0$ depending only on (X, ω) such that $\forall\psi \in \mathrm{psh}(X, \omega)$,

$$\int_X e^{-\alpha(\psi - \sup_X \psi)}\omega^n \le C.$$

By the above theorem, there exists an $\alpha > 0$ such that

$$\int_X e^{-\alpha\varphi}\omega^n \le C, \quad \int_X e^{-\alpha\psi}\omega^n \le C.$$

Using the previous theorem with $G = F \log \sqrt{F^2+1}$, let ψ be the unique function with $\sup_X \psi = 0$ and ψ solving

$$(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{G+b}\omega^n = \frac{e^F\sqrt{F^2+1}\omega^n}{\int_X e^F\sqrt{F^2+1}\omega^n}.$$

Tian's α-invariant

Given (X, ω) a complex manifold, there exist constants $\alpha > 0$ and $C > 0$ depending only on (X, ω) such that $\forall \psi \in \mathrm{psh}(X, \omega)$,

$$\int_X e^{-\alpha(\psi - \sup_X \psi)}\omega^n \le C.$$

By the above theorem, there exists an $\alpha > 0$ such that

$$\int_X e^{-\alpha\varphi}\omega^n \le C, \quad \int_X e^{-\alpha\psi}\omega^n \le C.$$

Using the previous theorem with $G = F \log \sqrt{F^2+1}$, let ψ be the unique function with $\sup_X \psi = 0$ and ψ solving

$$(\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{G+b}\omega^n = \frac{e^F\sqrt{F^2+1}\omega^n}{\int_X e^F\sqrt{F^2+1}\omega^n}.$$

Tian's α-invariant

Given (X, ω) a complex manifold, there exist constants $\alpha > 0$ and $C > 0$ depending only on (X, ω) such that $\forall \psi \in \mathrm{psh}(X, \omega)$,

$$\int_X e^{-\alpha(\psi - \sup_X \psi)}\omega^n \le C.$$

By the above theorem, there exists an $\alpha > 0$ such that

$$\int_X e^{-\alpha\varphi}\omega^n \le C, \quad \int_X e^{-\alpha\psi}\omega^n \le C.$$

Define $Q := e^{\delta(F+\varepsilon\psi-\lambda\varphi)}$ for $\delta, \varepsilon, \lambda > 0$ to be determined and assume that Q achieves a maximum at point p and let $B(p)$ be a coordinate ball.

For $\theta > 0$ to be determined, let $\eta(x) \in [1-\theta, 1]$ be a smooth cut-off function on X such that

$$\eta(p) = 1,\ \eta|_{\partial B} = 1-\theta,\ |\partial\eta|^2_\omega = O(\theta^2),\ |\nabla^2\eta|_\omega = O(\theta).$$

It can be shown that for λ large, $2n\delta\lambda = \alpha$ and $\theta << \delta$, we have

$$e^{-\delta(F+\varepsilon\psi-\lambda\varphi)}\Delta_\varphi(Q\eta) \geq \delta\eta(-C+\varepsilon n(\sqrt{F^2+1})^{1/n} I_F^{-1/n}),$$

where $I_F = \int_X e^F\sqrt{F^2+1}\,\omega^n$.

Applying the Alexandrov-Bakelman-Pucci (ABP) maximum principle to $Q\eta$, we get

$$\sup_B Q\eta \leq \sup_{\partial B} Q\eta + C_n\Big(\int_B \delta Q^{2n} e^{2F} ((-C + \varepsilon n(\sqrt{F^2+1})^{1/n} I_F^{-1/n})^-)^{2n}\omega^n \Big)^{1/2n}.$$

By the positivity of $(\sqrt{F^2+1})^{1/n}$ and $I_F^{-1/n}$, we find that the integral on the right-hand side is bounded above by

$$\int_{B\cap\{F\leq C\}} C\delta e^{2n\delta(F+\varepsilon\psi-\lambda\varphi)} e^{2F}\omega^n \leq C \int_{B\cap\{F\leq C\}} e^{2n\delta(\varepsilon\psi-\lambda\varphi)}\omega^n$$
$$\leq C\int_X e^{-2n\delta\lambda\varphi}\omega^n = C\int_X e^{-\alpha\varphi}\omega^n \leq C.$$

since $\psi \leq 0$ and by applying and using the property of the α-invariant, where C depends on ε and I_F.

This gives us that

$$Q(p) = \sup_X Q \leq (1-\theta) \sup_X Q + C \Rightarrow F + \varepsilon\psi - \lambda\varphi \leq C,$$

and so we can prove an upper bound for F if we can bound φ and ψ.

Using the Hölder continuity of the complex Monge-Ampère equation for Hermitian metrics by Dinew-Kolodziej and Blocki, it follows that φ and ψ are uniformly bounded.

A lower bound for F follows by a simple maximum principle argument and since I_F can be bounded above by the entropy, the dependency gets passed over. $\square$

Step 2: C^1 estimate on φ

Lemma

Let (φ, F) be a smooth solution to CCSC. Then there exists a constant C depending only on (X, ω) and the entropy such that

$$|\partial\varphi|^2_\omega \leq C.$$

Proof: Consider the quantity $Q := e^{-(F+\lambda\varphi)+\frac{1}{2}\varphi^2}(|\partial\varphi|^2_\omega + 1)$ and let $A := -(F + \lambda\varphi) + \frac{1}{2}\varphi^2$. We will compute $\Delta_\varphi Q$ for $\lambda > 0$ to be determined.

Step 2: C^1 estimate on φ cont.

Computing, we get that

$$
\begin{aligned}
&e^{-A}\Delta_\varphi Q \\
&= |\partial A|^2_{\omega_\varphi}(|\partial\varphi|^2_\omega + 1) + g^{k\bar\ell}{g_\varphi}^{i\bar j}\varphi_k R_{i\bar j}{}^{\bar r}{}_{\bar\ell}\varphi_{\bar r} - 2(\lambda-\varphi)|\partial\varphi|^2_{\omega_\varphi} \\
&\quad + \left(\underline{\mathrm{R}} - (\lambda-\varphi)n + {g_\varphi}^{i\bar j}((\lambda-\varphi)g_{i\bar j} - R_{i\bar j}) + |\partial\varphi|^2_{\omega_\varphi}\right)(|\partial\varphi|^2_\omega + 1) \\
&\quad - 2\mathrm{Re}(g^{k\bar\ell}A_k\varphi_{\bar\ell}) \color{red}{+ 2\mathrm{Re}({g_\varphi}^{i\bar j}g^{k\bar\ell}T^r_{ki}\varphi_{r\bar j}\varphi_{\bar\ell})} + {g_\varphi}^{i\bar j}g^{k\bar\ell}\varphi_{ki}\varphi_{\bar\ell\bar j} \\
&\quad + 2\mathrm{Re}({g_\varphi}^{i\bar j}A_i g^{k\bar\ell}(\varphi_k\varphi_{\bar\ell\bar j} + \varphi_{k\bar j}\varphi_{\bar\ell})) + {g_\varphi}^{i\bar j}g^{k\bar\ell}\varphi_{k\bar j}\varphi_{\bar\ell i} \\
&\geq -C(\lambda)(|\partial\varphi|^2_\omega + 1) + C(\lambda)\,\mathrm{tr}_{\omega_\varphi}\omega(|\partial\varphi|^2_\omega + 1) + |\partial\varphi|^2_{\omega_\varphi}|\partial\varphi|^2_\omega \\
&\quad \color{red}{+ 2\mathrm{Re}(g^{k\bar\ell}T^i_{ki}\varphi_{\bar\ell}) - 2\mathrm{Re}({g_\varphi}^{i\bar j}g^{k\bar\ell}g_{r\bar j}T^r_{ki}\varphi_{\bar\ell})},
\end{aligned}
$$

where the non-Kähler terms appear in red.

Since

$$
\begin{aligned}
&2\mathrm{Re}(g^{k\bar{\ell}}T^i_{ki}\varphi_{\bar{\ell}}) - 2\mathrm{Re}({g_\varphi}^{i\bar{j}}g^{k\bar{\ell}}g_{r\bar{j}}T^r_{ki}\varphi_{\bar{\ell}}) \\
&\geq -|\partial\varphi|^2_\omega - |T^i_{\cdot i}|^2_\omega - \mathrm{tr}_{\omega_\varphi}\omega|T^r_{ki}|^2_\omega - \mathrm{tr}_{\omega_\varphi}\omega|\partial\varphi|^2_\omega \\
&\geq -C(|\partial\varphi|^2_\omega + 1) - C\,\mathrm{tr}_{\omega_\varphi}\omega(|\partial\varphi|^2_\omega + 1),
\end{aligned}
$$

we can choose λ sufficiently large to get that

$$
e^{-A}\Delta_\varphi Q \geq -C(|\partial\varphi|^2_\omega + 1) + C\,\mathrm{tr}_{\omega_\varphi}\omega(|\partial\varphi|^2_\omega + 1) + |\partial\varphi|^2_{\omega_\varphi}|\partial\varphi|^2_\omega.
$$

An elementary consequence of the fact that $e^F = \frac{\omega_\varphi^n}{\omega^n}$ gives the following inequality

$$|\partial\varphi|^2_{\omega_\varphi}|\partial\varphi|^2_\omega + |\partial\varphi|^2_\omega \operatorname{tr}_{\omega_\varphi}\omega \geq \frac{1}{n-1}(|\partial\varphi|^2_\omega)^{1+\frac{1}{n}}e^{-\frac{F}{n}}.$$

At a maximum of Q, we have that

$$0 \geq e^{-A}\Delta_\varphi Q \geq (|\partial\varphi|^2_\omega)^{1+\frac{1}{n}}e^{-\frac{F}{n}} - C(|\partial\varphi|^2_\omega + 1).$$

Since we have bounds on F depending on (X, ω) and the entropy, we arrive at the desired upper bound on $|\partial\varphi|^2_\omega$. $\square$

Step 3: L^p estimate on $\mathrm{tr}_\omega \omega_\varphi$

Lemma

Let (φ, F) be a smooth solution to CCSC. For any $p > 0$, there exists a constant $C(p)$ depending only on p, (X, ω) and the entropy such that

$$\int_X (\mathrm{tr}_\omega \omega_\varphi)^p \omega^n \leq C(p).$$

Proof: Define $Q := e^{-\alpha(F+\lambda\varphi)}(\mathrm{tr}_\omega \omega_\varphi + 1)$ and let $A := -\alpha(F + \lambda\varphi)$ where $\alpha, \lambda > 0$ are to be determined. We first compute:

$$e^{-A}\Delta_\varphi Q = (\Delta_\varphi A + |\partial A|^2_{\omega_\varphi})(\mathrm{tr}_\omega \omega_\varphi + 1) + \Delta_\varphi \mathrm{tr}_\omega \omega_\varphi$$
$$+ 2\mathrm{Re}({g_\varphi}^{i\bar{j}} A_i \partial_{\bar{j}} \mathrm{tr}_\omega \omega_\varphi).$$

Using an inequality due to Cherrier to bound the second term

$$
\begin{aligned}
e^{-A}\Delta_\varphi Q
&\geq (\Delta_\varphi A + |\partial A|^2_{\omega_\varphi})(\operatorname{tr}_\omega \omega_\varphi + 1) + \frac{2}{\operatorname{tr}_\omega \omega_\varphi}\operatorname{Re}({g_\varphi}^{i\bar{j}} T^k_{ki}\partial_{\bar{j}} \operatorname{tr}_\omega \omega_\varphi) \\
&\quad + \Delta F \; - C \operatorname{tr}_\omega \omega_\varphi \operatorname{tr}_{\omega_\varphi} \omega + \frac{|\partial \operatorname{tr}_\omega \omega_\varphi|^2_{\omega_\varphi}}{\operatorname{tr}_\omega \omega_\varphi} + 2\operatorname{Re}({g_\varphi}^{i\bar{j}} A_i \partial_{\bar{j}} \operatorname{tr}_\omega \omega_\varphi) \\
&\geq \tilde{\Delta A}(\operatorname{tr}_\omega \omega_\varphi + 1) + |\partial A|^2_{\omega_\varphi} + \Delta F - C \operatorname{tr}_\omega \omega_\varphi \operatorname{tr}_{\omega_\varphi} \omega \\
&\quad - \frac{{g_\varphi}^{i\bar{j}} T^k_{ki} \overline{T^\ell_{\ell j}}}{\operatorname{tr}_\omega \omega_\varphi} - 2\operatorname{Re}({g_\varphi}^{i\bar{j}} A_i \overline{T^\ell_{\ell \bar{j}}})
\end{aligned}
$$

where we used the completed square:

$$
\begin{aligned}
0 &\leq \frac{1}{\operatorname{tr}_\omega \omega_\varphi}{g_\varphi}^{i\bar{j}}(A_i \operatorname{tr}_\omega \omega_\varphi + T^k_{ki} + \partial_i \operatorname{tr}_\omega \omega_\varphi)(A_{\bar{j}} \operatorname{tr}_\omega \omega_\varphi + \overline{T^\ell_{\ell j}} + \partial_{\bar{j}} \operatorname{tr}_\omega \omega_\varphi) \\
&= |\partial A|^2_{\omega_\varphi} \operatorname{tr}_\omega \omega_\varphi + \frac{{g_\varphi}^{i\bar{j}} T^k_{ki} \overline{T^\ell_{\ell j}}}{\operatorname{tr}_\omega \omega_\varphi} + \frac{|\partial \operatorname{tr}_\omega \omega_\varphi|^2_{\omega_\varphi}}{\operatorname{tr}_\omega \omega_\varphi} + 2\operatorname{Re}({g_\varphi}^{i\bar{j}} A_i \overline{T^\ell_{\ell j}}) \\
&\quad + 2\operatorname{Re}({g_\varphi}^{i\bar{j}} A_i \partial_{\bar{j}} \operatorname{tr}_\omega \omega_\varphi) + \frac{2}{\operatorname{tr}_\omega \omega_\varphi}\operatorname{Re}({g_\varphi}^{i\bar{j}} T^k_{ki}\partial_{\bar{j}} \operatorname{tr}_\omega \omega_\varphi).
\end{aligned}
$$

Now, we can bound the non-Kähler terms as follows:

$$
\begin{aligned}
-\frac{{g_\varphi}^{i\bar{j}} T^k_{ki}\overline{T^\ell_{\ell j}}}{\operatorname{tr}_\omega \omega_\varphi} - 2\mathrm{Re}({g_\varphi}^{i\bar{j}} A_i \overline{T^\ell_{\ell\bar{j}}}) &\geq -|T^k_{k\cdot}|^2_{\omega_\varphi} - |\partial A|^2_{\omega_\varphi} - |T^k_{k\cdot}|^2_{\omega_\varphi} \\
&\geq -C \operatorname{tr}_{\omega_\varphi} \omega - |\partial A|^2_{\omega_\varphi} \\
&\geq -C \operatorname{tr}_{\omega_\varphi} \omega \operatorname{tr}_\omega \omega_\varphi - |\partial A|^2_{\omega_\varphi}
\end{aligned}
$$

where we used the fact that we have a lower bound on $\operatorname{tr}_\omega \omega_\varphi$ from the arithmetic-geometric mean inequality.

Choosing λ sufficiently large, we have

$$
e^{-A}\Delta_\varphi Q \geq \alpha(-C(\lambda) + C(\lambda) \operatorname{tr}_{\omega_\varphi} \omega)(\operatorname{tr}_\omega \omega_\varphi + 1) + \Delta F
$$

Integrating the following inequality with respect to ω_φ^n

$$\begin{aligned}\frac{1}{2p+1}\Delta_\varphi(Q^{2p+1}) &= 2pQ^{2p-1}|\partial Q|^2_{\omega_\varphi} + Q^{2p}\Delta_\varphi Q \\ &\geq 2pQ^{2p-2}|\partial Q|^2_\omega e^A + Q^{2p}\Delta_\varphi Q,\end{aligned}$$

and using that $\omega_\varphi^n = e^F\omega^n$, we have that

$$\begin{aligned}\int_X 2pe^{A+F}Q^{2p-2}|\partial Q|^2_\omega\omega^n + \int_X \alpha e^{A+F}C(\lambda)\operatorname{tr}_{\omega_\varphi}\omega(\operatorname{tr}_\omega\omega_\varphi + 1)Q^{2p}\omega^n \\ + \int_X e^{A+F}Q^{2p}\Delta F\omega^n \leq \int_X \alpha e^{A+F}C(\lambda)(\operatorname{tr}_\omega\omega_\varphi + 1)Q^{2p}\omega^n.\end{aligned}$$

$$
\begin{aligned}
&\int_X e^{A+F} Q^{2p} \Delta F \omega^n \\
&\quad \geq \int_X e^{A+F} Q^{2p} \sqrt{-1}((\alpha - 1)\partial F + \alpha\lambda\partial\varphi) \wedge \bar{\partial} F \wedge \omega^{n-1} \\
&\qquad - \int_X e^{A+F} 2p Q^{2p-1} \sqrt{-1} \partial Q \wedge \bar{\partial} F \wedge \omega^{n-1} \\
&\qquad - C \int_X e^{A+F} Q^{2p} |\partial F|_\omega \omega^n \\
&\quad \geq \int_X C(\alpha) Q^{2p} e^{A+F} |\partial F|_\omega^2 - C(\alpha, \lambda) e^{A+F} Q^{2p} \omega^n \\
&\qquad - \int_X p e^{A+F} Q^{2p-2} |\partial Q|_\omega^2 \omega^n
\end{aligned}
$$

Choosing α, λ sufficiently large and substituting back into our original integral inequality, we arrive at

$$\int_X (\mathrm{tr}_\omega\, \omega_\varphi)^{2p+1+\frac{1}{n-1}} \omega^n \leq C \int_X (\mathrm{tr}_\omega\, \omega_\varphi)^{2p+1} \omega^n$$

where $p = 0$ is the base case for the iteration and is bounded:

$$\int_X (\mathrm{tr}_\omega\, \omega_\varphi)^{1+\frac{1}{n-1}} \omega^n \leq C \int_X \mathrm{tr}_\omega\, \omega_\varphi\, \omega^n \leq C vol(X).$$

□

Step 4: L^∞ bound on $\operatorname{tr}_\omega \omega_\varphi$

Lemma

Let (φ, F) be a smooth solution to CCSC. Then there exists a constant C depending only on (X, ω) and the entropy such that

$$\max_X(\operatorname{tr}_\omega \omega_\varphi) + \max_X |\partial F|^2_{\omega_\varphi} \leq C.$$

Proof: We will apply the maximum principle to

$$Q := e^{A(F)}|\partial F|^2_{\omega_\varphi} + N(\operatorname{tr}_\omega \omega_\varphi)^{B+1}$$

for $A(F)$ a real-valued function and $B, N \in \mathbb{N}$ to be determined.

For the first term, we have

$$
\begin{aligned}
& e^{-A(F)}\Delta_\varphi(e^{A(F)}|\partial F|^2_{\omega_\varphi}) \\
& \geq 2\mathrm{Re}(g_\varphi{}^{p\bar{q}}(\Delta_\varphi F)_p F_{\bar{q}}) + 2\mathrm{Re}(g_\varphi{}^{i\bar{j}} g_\varphi{}^{p\bar{q}} g_\varphi{}^{r\bar{k}} \tilde{T}_{pi\bar{k}} \tilde{\nabla}_r F_{\bar{j}} F_{\bar{q}}) \\
& \quad + g_\varphi{}^{i\bar{j}} g_\varphi{}^{p\bar{q}} g_\varphi{}^{t\bar{r}} \tilde{\nabla}_i (\overline{\tilde{T}_{qj\bar{t}}}) F_{\bar{r}} F_p + g_\varphi{}^{p\bar{q}} g_\varphi{}^{k\bar{\ell}} R_{k\bar{q}} F_{\bar{\ell}} F_p \\
& \quad - g_\varphi{}^{p\bar{q}} g_\varphi{}^{k\bar{\ell}} g_\varphi{}^{r\bar{s}} \tilde{\nabla}_{\bar{q}} (\tilde{T}_{rk\bar{s}}) F_{\bar{\ell}} F_p + (1 - (A' - \frac{1}{2}))|\tilde{\nabla}\bar{\tilde{\nabla}} F|^2_{\omega_\varphi} \\
& \quad + (A'' - (A' - \frac{1}{2}))|\partial F|^4_{\omega_\varphi} + A'\Delta_\varphi F|\partial F|^2_{\omega_\varphi}
\end{aligned}
$$

To control the non-Kähler terms, we choose $A(F)$ such that

$$
1 - (A' - \frac{1}{2}) \geq \frac{1}{2} \text{ and } A'' - (A' - \frac{1}{2}) = \varepsilon > 0
$$

which we can accomplish with $A' = \kappa e^F + \frac{1}{2} - \varepsilon$ and $A'' = \kappa e^F$ and κ, ε chosen sufficiently small. In the Kähler case, it suffices to choose $A' = \frac{1}{2}$.

Expanding terms, applying several instances of Young's inequality and choosing B sufficiently large, we arrive at

$$e^{-A(F)}\Delta_\varphi(e^{A(F)}|\partial F|^2_{\omega_\varphi}) \geq -C(\mathrm{tr}_\omega\,\omega_\varphi)^B g^{i\bar{j}} {g_\varphi}^{k\bar{\ell}} {g_\varphi}^{p\bar{q}} \partial_i {g_\varphi}_{k\bar{q}} \partial_{\bar{j}} {g_\varphi}_{p\bar{\ell}} - C(\mathrm{tr}_\omega\,\omega_\varphi)^B|\partial F|^2_{\omega_\varphi} + \frac{1}{4}|\tilde{\nabla}\bar{\tilde{\nabla}} F|^2_{\omega_\varphi}.$$

We now use the second term of Q to bound the blue term above, since, by a computation by Cherrier, we have

$$\begin{aligned}\Delta_\varphi\,\mathrm{tr}_\omega\,\omega_\varphi = {} & {g_\varphi}^{p\bar{j}} {g_\varphi}^{i\bar{q}} g^{k\bar{\ell}} \nabla_k {g_\varphi}_{i\bar{j}} \nabla_{\bar{\ell}} {g_\varphi}_{p\bar{q}} + 2\mathrm{Re}({g_\varphi}^{i\bar{j}} g^{k\bar{\ell}} T^p_{ki} \nabla_{\bar{\ell}} {g_\varphi}_{p\bar{j}}) \\ & + {g_\varphi}^{i\bar{j}} g^{k\bar{\ell}} T^p_{ik} \overline{T^q_{j\ell}} {g_\varphi}_{p\bar{q}} + g^{i\bar{j}} F_{i\bar{j}} - \mathrm{R} + {g_\varphi}^{i\bar{j}} \nabla_i \overline{T^\ell_{j\ell}} \\ & + {g_\varphi}^{i\bar{j}} g^{k\bar{\ell}} \nabla_{\bar{\ell}} T^p_{ik} - {g_\varphi}^{i\bar{j}} g^{k\bar{\ell}} {g_\varphi}_{k\bar{q}} (\nabla_i \overline{T^q_{j\ell}} - R_{i\bar{\ell}p\bar{j}} g^{p\bar{q}}) \\ & - {g_\varphi}^{i\bar{j}} g^{k\bar{\ell}} T^p_{ik} \overline{T^q_{j\ell}} g_{p\bar{q}}\end{aligned}$$

We will use the fact that

$$g^{i\bar{j}}F_{i\bar{j}} \geq -\frac{\frac{|\tilde{\nabla}\bar{\tilde{\nabla}}F|^2_{\omega_\varphi}}{4}}{e^{-A(F)}N(B+1)(\mathrm{tr}_\omega\,\omega_\varphi)^B} - C\delta(\mathrm{tr}_\omega\,\omega_\varphi)^2,$$

and after converting the covariant derivatives into partial derivatives, we have

$$\Delta_\varphi \mathrm{tr}_\omega\,\omega_\varphi \geq (1-\varepsilon){g_\varphi}^{p\bar{j}}{g_\varphi}^{i\bar{q}}g^{k\bar{\ell}}\partial_k {g_\varphi}_{i\bar{j}}\partial_{\bar{\ell}}{g_\varphi}_{p\bar{q}} - C(\mathrm{tr}_\omega\,\omega_\varphi)^B$$
$$-\frac{\frac{|\tilde{\nabla}\bar{\tilde{\nabla}}F|^2_{\omega_\varphi}}{4}}{e^{-A(F)}N(B+1)(\mathrm{tr}_\omega\,\omega_\varphi)^B}.$$

Observe that

$$\frac{1}{B+1}\Delta_\varphi(\mathrm{tr}_\omega\,\omega_\varphi)^{B+1} = B(\mathrm{tr}_\omega\,\omega_\varphi)^{B-1}|\partial\,\mathrm{tr}_\omega\,\omega_\varphi|^2_{\omega_\varphi} + (\mathrm{tr}_\omega\,\omega_\varphi)^B\Delta_\varphi \mathrm{tr}_\omega\,\omega_\varphi$$
$$\geq (\mathrm{tr}_\omega\,\omega_\varphi)^B\Delta_\varphi \mathrm{tr}_\omega\,\omega_\varphi.$$

From this, we get that

$$\begin{aligned}\Delta_\varphi Q &= \Delta_\varphi(e^{A(F)}|\partial F|^2_{\omega_\varphi} + N(\mathrm{tr}_\omega\,\omega_\varphi)^{B+1})\\ &\geq -C(\mathrm{tr}_\omega\,\omega_\varphi)^B|\partial F|^2_{\omega_\varphi} - C(\mathrm{tr}_\omega\,\omega_\varphi)^{2B}\\ &\geq -C(\mathrm{tr}_\omega\,\omega_\varphi)^B Q.\end{aligned}$$

We can now prove the L^∞ bound using Moser iteration and Hölder and Sobolev inequalities with the base case:

$$\int_X |\partial F|^2_{\omega_\varphi}\omega^n = \int_X F\Delta_\varphi F\omega_\varphi^n \leq C\int_X (1+(\mathrm{tr}_\omega\,\omega_\varphi)^{n-1})\omega^n \leq C$$
$$\text{and} \int_X (\mathrm{tr}_\omega\,\omega_\varphi)^{B+1}\omega^n \leq C$$

where C depends on (X, ω) and the entropy. □

Open Problems

- Specific continuity path and openness for CCSC metrics
- Understanding geometric interpretation of bounded entropy quantity
- Obstructions to existence of CCSC metrics
- Existence of neighboring CCSC metrics
- Generalizing Chern-Calabi flow to a broader class of complex manifolds

Thank you!