Compact Kähler threefolds with nef anticanonical line bundle

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Nef line bundle

Nef line bundle, [DPS94]

Let (X, ω) be a compact Kähler manifold. A line bundle L over X is nef if for every $\varepsilon > 0$ there exists a smooth hermitian metric h_ε on L such that the Chern curvature satisfies

$$\Theta(h_\varepsilon) \geqslant -\varepsilon\omega.$$

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Algebraic description

If C is a curve in X and L is nef, $(L \cdot C) \geqslant 0$. It is an equivalent definition if X is projective, but not in general (e.g. $L = \mathcal{O}(-(n-1)E) = -K_X$ where E is exceptional divisor of blow-up X of a point in a very general torus of dimension n).

Nef vector bundle

A vector bundle E over a compact Kähler manifold is nef if and only if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef.

Theorem, Păun 98

Let L be a line bundle over a compact Kähler manifold. Then L is nef if and only if $L|_Z$ is a psef line bundle when restriction to any (irreducible) closed analytic subset.

Ture for vector bundle with the definition of (strongly) pseudoeffective vector bundle given in [BDPP] (Wu22).
Conjecturally, K_X is nef iff $(K_X \cdot C) \geqslant 0$ for any curve C.

Background

Certain “positively curved” varieties, which are often formulated to have positive holomorphic bisectional curvatures/ nefness/ pseudoeffectivity, tangent bundles, or anticanonical divisors, have occupied an important place in the classification theory.
The famous work of Siu-Yau and Mori have given a beautiful characterization of projective spaces, in terms of positive holomorphic bisectional curvatures or ample tangent bundles.
Since that time, it has become clear that structures of positively curved varieties are closely related to the geometry of rational curves and have a certain rigidity.

Background

One of the central problems in this field is to understand structures reflecting rational curves and rigidities, by using naturally associated fibrations, such as Albanese maps, Iitaka fibrations, and maximal rationally connected (for short, MRC) fibrations.
In some sense, these fibrations decompose the manifold into building blocks of different curvature nature.

compact Kähler manifold with nef anticanonical line bundle

Conjecture

Let X be a compact Kähler manifold with the nef anti-canonical bundle $-K_X$. Then, there exists a fibration $\varphi : X \to Y$ with the following:

- $\varphi : X \to Y$ is a locally trivial fibration;
- Y is a compact Kähler manifold with $c_1(Y) = 0$;
- F, which is the fiber of $\varphi : X \to Y$, is rationally connected.

Known in projective case by Cao-Höring.
But widely open in the compact Kähler case.

Examples

By Beauville-Bogomolov decomposition theorem and the above conjecture, the building blocks of compact Kähler manifold with nef anticanonical line bundle are

- torus
- strict Calabi-Yau manifold (i.e. simply connected, $K_X = \mathcal{O}_X$)
- holomorphic symplectic manifold
- rational connected with nef anticanonical line bundle (e.g. Fano)

Main result

Theorem, Matsumura-Wu 23

Let X be a non-projective compact Kähler 3-fold with nef anti-canonical bundle. Then X admits a finite étale cover that is one of the following:

- a compact Kähler manifold with vanishing first Chern class;
- the product of a K3 surface and the projective line $\mathbb{P}^1$;
- the projective space bundle $\mathbb{P}(E)$ of a numerical flat vector bundle E of rank 2 over a 2-dimensional (compact complex) torus.

This implies the structure conjecture in the case of $\dim X = 3$.

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Arguments of Cao-Höring

For simplicity, we suppose that the MRC quotient $\varphi : X \dashrightarrow R(X)$ is a holomorphic map onto a smooth projective variety $R(X)$.

(1) Construct a φ-ample line bundle B on X such that the direct image sheaf $\varphi_*(pB) := \varphi_*\mathcal{O}_X(pB)$ is weakly positively curved and satisfies that $c_1(\varphi_*(pB)) = 0$ for $1 \ll p \in \mathbb{Z}$.

(2) Show that $\varphi_*(pB)$ is numerical flat.

(3) Simpson's result shows that $\varphi_*(pB)$ admits a flat connection, which implies that $X \to R(X)$ is a locally trivial fibration.

Arguments of Cao-Höring

A line bundle B is called $\varphi-$ample if there exists a Kähler form $\omega_{R(X)}$ on $R(X)$ such that $c_1(B) + \varphi^*\omega_{R(X)}$ is a Kähler class on X.
$\varphi_*(pB)$ is numerical flat means that $\varphi_*(pB)$, $\varphi_*(pB)^*$ are nef vector bundles. (Hence semistable with trivial c_1, c_2 and hence flat by Simpson's correspondence.)
Let A is a very ample line bundle on X. B will be of form like $A - \frac{1}{\mathrm{rank}(\varphi_* A)}\det(\varphi_* A)$ in Cao-Höring.
In our case, B will be $-mK_X - \frac{1}{\mathrm{rank}(\varphi_*(-mK_X))}\det(\varphi_*(-mK_X))$ for $m \gg 0$ since $-K_X$ is $\varphi-$ample and fibers of φ are rational curves.

Difficulty

The MRC quotient a priori is just meromorphic.
How to show $-K_X$ is relative ample?

Idea

In 3-dim compact Kähler non-projective case, the MMP consists of only one step of Mori fiber (for short, MF) space such that $-K_X$ is relative ample.

In general, MMP involves singular space...
Example (Iitaka): Let X be any resoultion of singularity of $A/\langle\pm 1\rangle$ (A 3 dimension torus). There is no smooth manifold bimeromorphic to X with nef canonical divisor.

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Reduction

Let $R(X)$ be the MRC quotient of X.

Reduction to the case of $\dim R(X) = 2$

In the case of $\dim R(X) \leqslant 1$, a general fiber F of $\varphi : X \dashrightarrow R(X)$ is rationally connected, and hence has no (non-zero) holomorphic differential forms; therefore we have $h^2(X, \mathcal{O}_X) = h^0(X, \Omega_X^2) = 0$ by $\dim R(X) = 1$, which implies that X is projective.
In the case of $\dim R(X) = 3$, the manifold X is non-uniruled; hence K_X is pseudo-effective, which follows from [BDPP] for projective manifolds of any dimension and from [Brunella06] for compact Kähler manifolds of dimension $\leqslant 3$. This implies that $c_1(X) = c_1(K_X) = 0$ since $-K_X$ is nef.

3-dim compact Kähler MMP

Theorem, Höring-Peternell 16

Let X be a $\mathbb{Q}$-factorial compact Kähler space of dimension 3 with terminal singularities. Assume that $\dim R(X) = 2$, where $R(X)$ is the base of an MRC fibration $X \dashrightarrow R(X)$ of X. Then, we have that X is bimeromorphic to a MF (Mori fiber) space; more precisely, there exist a bimeromorphic map $\pi : X \dashrightarrow X'$ and a MF space $\varphi : X' \to S$ such that

(a) $X \dashrightarrow X'$ is obtained from the composition of divisorial contractions and flips;

(b) X' is a $\mathbb{Q}$-factorial compact Kähler space with terminal singularities;

(c) S is a $\mathbb{Q}$-factorial compact Kähler space of dimension 2 with klt singularities;

3-dim compact Kähler MMP

(d) S is non-uniruled and K_S is pseudo-effective;

(e) $-K_{X'}$ is φ-ample and the relative Picard number $\rho(X'/S)$ is 1;

(f) $\varphi : X' \to S$ is equi-dimensional and of relative dimension 1.

Key point

$-K_{X'}$ is φ-ample. Show first that X' has the desired structure and then that $X' \simeq X$ by calculation of intersection numbers as Peternell-Serrano.

Bad news: X' may be singular a priori... How to relate positivity of X' and S?

Bott-Chern cohomology

A Kähler spaces is a normal analytic variety admitting a Kähler form (i.e., a smooth positive $(1,1)$-form on X with local potential). Let X be a normal analytic variety. Then, the Bott-Chern cohomology group of X is defined by

$$H^{1,1}_{BC}(X, \mathbb{C}) := H^1(X, \mathbb{R}\mathcal{O}_X).$$

The first Chern class $c_1(L) \in H^{1,1}_{BC}(X, \mathbb{C})$ of a line bundle L on X is defined by the Bott-Chern cohomology class of $(\sqrt{-1}/2\pi)\Theta_h(L)$, where $\Theta_h(L)$ denotes the Chern curvature of a smooth metric h on L (e.g., which is constructed by a partition of unity).

Bott-Chern cohomology

Proposition (Boucksom-Eyssidieux-Guedj 13)

Let $\alpha \in H^{1,1}_{BC}(X, \mathbb{C})$ be a Bott-Chern cohomology class on a normal analytic variety X, and let T be a positive current on X_{reg} representing the restriction $\alpha|_{X_{\mathrm{reg}}} \in H^{1,1}_{BC}(X_{\mathrm{reg}}, \mathbb{C})$. Then, the current T is uniquely extended to the positive current with local potential on X representing $\alpha \in H^{1,1}_{BC}(X, \mathbb{C})$.

$\mathbb{Q}-$conic bundle

$\mathbb{Q}-$conic bundle (Mori-Prokhorov 08)

Let X and S be normal analytic varieties. A fibration $\varphi : X \to S$ is called *a* $\mathbb{Q}$*-conic bundle* if it satisfies following conditions:

- X has terminal singularities;
- $\varphi : X \to S$ is equi-dimensional and of relative dimension 1;
- $-K_X$ is φ-ample.

Discrimant divisor(Mori-Prokhorov 08)

The discriminant divisor Δ is defined by the union of divisorial components of the non-smooth locus $\{s \in S \mid \varphi \text{ is not a smooth fibration at s}\}$.

conic bundle

Conic bundle (Mori-Prokhorov 08)

Let $\varphi : X \to S$ be a $\mathbb{Q}$-conic bundle. The fibration $\varphi : X \to S$ is said to be a *conic bundle* if X and S are smooth.

Theorem, Matsumura-Wu 23

Let $\varphi : X \to S$ be a conic bundle with discriminant divisor Δ. Then, we have

$$\varphi_*(c_1(K_X)^2) = -4c_1(K_S) - c_1(\Delta) = \frac{4}{3}c_1(\varphi_*(-K_X)) + \frac{1}{3}c_1(\Delta).$$

The same formula holds for $\mathbb{Q}-$conic bundle by Proposition of BEG.

$\mathbb{Q}-$conic bundle

Toroidal case

A $\mathbb{Q}$-conic bundle $\varphi : X \to S$ is said to be *toroidal* at $s \in S$ with respect to $\mu_m := \mathbb{Z}/m\mathbb{Z}$ if X is isomorphic to the quotient of $\mathbb{P}^1 \times \mathbb{C}^2$ over a neighborhood of s by the μ_m-action defined by

$$(t; z_1, z_2) \to (\varepsilon^b t; \varepsilon z_1, \varepsilon^{-1} z_2),$$

where b is an integer with $\gcd(m, b) = 1$ and ε is a primitive m-th root of unity. Note that the singularities of X consists of cyclic quotient singularities of types $(1/m)(b, 1, -1)$ and $(1/m)(-b, 1, -1)$; furthermore, the singularities of the base $S \cong \mathbb{C}^2/\mu_m$ are the cyclic quotient of type A_{m-1}.

classification of 3-dim $\mathbb{Q}-$conic bundle

Mori-Prokhorov 08

Let $\varphi : X \to S$ be a 3-dimensional $\mathbb{Q}$-conic bundle and $\Delta \subset S$ be the discriminant divisor. Then $s \notin \Delta$ if and only if $\varphi : X \to S$ is toroidal at s.

Example (A global $\mathbb{Q}-$conic bundle)

For a Kummer surface $S := A/\mu_2$ with a torus A of dimension 2, we consider

$$X' := (\mathbb{P}^1 \times A)/\mu_2 \to S = A/\mu_2,$$

where μ_2 acts on $\mathbb{P}^1 \times A$ by $-1 \cdot (t, z_1, z_2) = (-t, -z_1, -z_2)$. Both S and X' are simply connected and $\varphi : X' \to S$ is a $\mathbb{Q}$-conic bundle such that $-K_{X'}$ is nef. However X' is not outcome of MMP for some smooth X with $-K_X$ nef (cf. Peternell-Serrano).

Consequence

Corollary, Matsumura-Wu 23

We consider the MF space $\varphi : X' = X_N \to S$ in 3-dim Kähler MMP. Then, we have:
(1) The Bott-Chern cohomology class $-4c_1(K_S) - c_1(\Delta)$ is pseudo-effective, where Δ is the discriminant divisor of the MF space $\varphi : X \to S$ (which is a $\mathbb{Q}$-conic bundle).
(2) The relation $\Delta = 0$ and $c_1(K_S) = 0$ holds; in particular, $\varphi : X' \to S$ is toroidal over S. Furthermore, when S are smooth, the variety X is automatically smooth and $\varphi : X' \to S$ is a (locally trivial) $\mathbb{P}^1$-bundle.

Technical Remark

transformation of Kähler class

An analytic variety is Kähler if there exists a Bott-Chern class whose representative is locally restriction of Kähler metric of ambiant space. Let $\pi : X \to X'$ be a modification of analytic variety. Let ω be a Kähler class on X. $\pi_* \omega$ does not define necessarily a Bott-Chern class (due to non-existence of local potential.)

Assume that π is a blow up with exceptional divisor E and ω' be a Kähler form on X'. We should consider $-K_X + \epsilon(\pi^* \omega' - \delta E)$ for $0 < \delta \ll 1$ such that $\pi_*(-K_X + \epsilon(\pi^* \omega' - \delta E)) = -K_{X'} + \epsilon \omega'$ defines a Bott-Chern class.

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End of proof

Let $\varphi : X' \to S$ be the MF space.

(1) Show that $\varphi_*(-pK_{X'})$ is weakly positively curved with trivial first Chern class for $1 \ll p$ by positivity of direct image.

(2) Show that $\varphi_*(-pK_{X'})$ is numerical flat orbifold vector bundle (by -23).

(3) By Campana04, S is either quotient of torus or normal K3. Show that $\varphi_*(pB)$ is trivial over some quasi-étale cover. Deduce a contradiction by intersection numbers if S is not smooth.

End of proof

In general, the positivity of direct image is insensible to singularity. It is conjectured that the fundamental group of the regular part of klt compact Kähler Calabi-Yau space is infinite if and only if it contains a torus factor in the singular Beauville-Bogomolov decomposition theorem. If this holds, Step 3 should be able to generalise to high dimensional case.

Thank you for your attention!