Constant curvature conical metrics

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Joint works with Rafe Mazzeo, Bin Xu and Misha Karphukhin

Outline

Constant curvature metrics on Riemann surfaces

- Classical uniformization theorem: for a given Riemann surface, there is a unique (smooth) constant curvature metric

$$\text{(Gauss–Bonnet)} \qquad \chi(\Sigma) \qquad = \frac{1}{2\pi} KA$$

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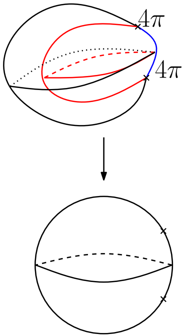
- Near a cone point with angle $2\pi\beta$, in geodesic polar coordinates

$$g = \begin{cases} dr^2 + \beta^2 r^2 d\theta^2 & K = 0 \quad \text{(flat)} \\ dr^2 + \beta^2 \sin^2 r d\theta^2 & K = 1 \quad \text{(spherical)} \\ dr^2 + \beta^2 \sinh^2 r d\theta^2 & K = -1 \quad \text{(hyperbolic)} \end{cases}$$

- In conformal coordinates $z = (\beta r)^{1/\beta} e^{i\theta}$, $\quad g = f(z)|z|^{2(\beta-1)}|dz|^2$

Some examples of spherical conical metrics



Branched covers
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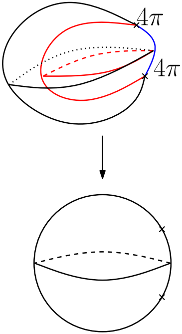

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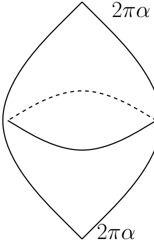

Spherical footballs

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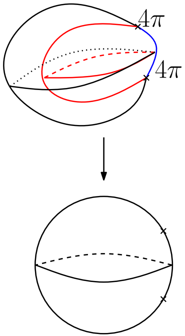

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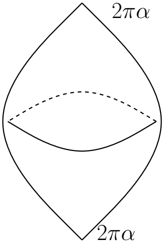

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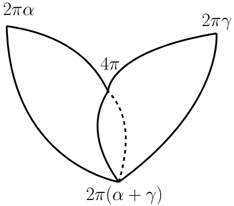

“Heart”: footballs glued along geodesics

The study of constant curvature conical metrics is related to:

- Magnetic vortices: solitons of gauged sigma-models on a Riemann surface
- Mean Field Equations: models of electro-magnetism
- Toda system: multi-dimensional version
- Higher dimensional analogue: Kähler–Einstein metrics with conical singularities
- Hyperbolic conical metrics: bridges between pointed and unpointed Riemann moduli spaces

This subject can be approached in many ways:

- PDE: singular Liouville equations
- Complex analysis: developing maps and Schwarzian derivatives
- Synthetic geometry: cut-and-glue

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A singular uniformization problem

Consider the following "conical data":

- n distinct points $\mathfrak{p} = (p_1, \ldots, p_n)$
- Angle data $\vec{\beta} = (\beta_1, \ldots, \beta_n), \ \beta_i \in \mathbb{R}^+ \setminus \{1\}$
- Conformal structure $\mathfrak{c}$ given by the underlying Riemann surface

Question

Given conical data $(\mathfrak{p}, \vec{\beta}, \mathfrak{c})$, does there exist a unique constant curvature conical metric with this data?

When uniformization holds

Theorem (Heins '62, McOwen '88, Troyanov '91, Luo–Tian '92)

For any compact Riemann surface $(M, \mathfrak{c})$ *and conical data* $(\mathfrak{p}, \vec{\beta})$ *with*

- $\chi(M, \vec{\beta}) \leq 0$; *or*
- $\chi(M, \vec{\beta}) > 0$, $\vec{\beta} \in T \subset (0,1)^k$

there is a unique constant curvature conical metric with this data.

Theorem (Mazzeo–Weiss '15)

If $\vec{\beta} \in (0,1)^k$, *then there is a well-defined* $(6\gamma - 6 + 3k)$*-dimensional moduli space.*

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Spherical metrics with large cone angles

- The remaining case: $\chi(M, \vec{\beta}) > 0$, at least one of the angles greater than 2π
- Uniformization fails in this case
- Existence: constraints on conical data $(\mathfrak{p}, \vec{\beta}, \mathfrak{c})$
  Mondello–Panov '16, Chen–Lin '17, Chen–Kuo–Lin–Wang '18...
- Uniqueness: usually fails
  Chen–Wang–Wu–Xu '14, Eremenko '17,
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- Deformation: obstructions exist [Z '19]
- Literature: Troyanov '91,
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Outline of the main result

Our results provide new understanding of the local structure of the moduli space where it is not smoothly parametrized:

Theorem (Mazzeo–Z '19)

- *The local deformation with respect to* $(\mathfrak{c}, \mathfrak{p}, \vec{\beta})$ *has rigidity precisely when* $2 \in \mathrm{Spec}(\Delta_g^{\mathrm{Fr}})$*;*
- *It can be "desingularized" by adding more coordinates via splitting of cone points.*

- Understanding this problem through a nonlinear PDE:

$$\{\text{Constant curvature } K \text{ conical metrics}\}$$
$$\updownarrow$$
$$\left\{ \begin{array}{c} \text{Solutions to the Liouville equation} \\ \Delta_{g_0} u - K e^{2u} + K_{g_0} = 0 \end{array} \right\}$$

Here g_0 is either a smooth metric (then u has singularities); or a conical metric with the given conical data (then u is bounded).

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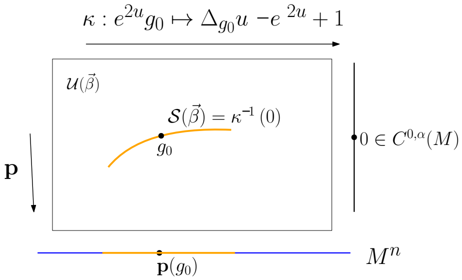

- Fix $\vec{\beta}$ for simplicity
- $\mathcal{U}(\vec{\beta})$: total space of conical metrics with angles $\vec{\beta}$
- $\mathcal{S}(\vec{\beta})$: space of spherical conical metrics with angles $\vec{\beta}$
- κ: curvature map
- **p**: map to the cone positions
- The linearized operator is $\Delta_{g_0} - 2$
- When $2 \notin \mathrm{Spec}(\Delta_{g_0}^{\mathrm{Fr}})$, implicit function theorem applies to get a neighborhood of $\mathbf{p}(g_0)$
- Otherwise we have a problem

Deformation and linear obstructions

- Fix a spherical conical metric $g_0 \in \mathcal{S}(\vec{\beta})$. We study local deformations $g_t : (-\epsilon, \epsilon) \to \mathcal{S}(\vec{\beta})$ and cone point positions $\mathfrak{p}_t = \mathbf{p}(g_t)$.
- We have $g_t = e^{2u_t} g_0$ where u_t solve the singular Liouville equation

$$\Delta_{g_0} u_t - e^{2u_t} + 1 = 0$$

Linearized equation: $(\Delta_{g_0} - 2)v = 0$ where $v := \partial_t u_t|_{t=0}$

- If $v \in \ker(\Delta_{g_0}^{\mathrm{Fr}} - 2)$ where $\Delta_{g_0}^{\mathrm{Fr}}$ is the Friedrichs Laplacian, then $\partial_t \mathfrak{p}_t|_{t=0} = 0$: obstruction to injectivity of $\mathbf{p}$.
- $\partial_t \mathfrak{p}_t|_{t=0}$ gives the singular terms of v (those not in the Friedrichs domain). If $\ker(\Delta_{g_0}^{\mathrm{Fr}} - 2) \neq 0$ then it might be impossible to find a solution with given singular terms: obstruction to surjectivity of $\mathbf{p}$.
- We say $\vec{A}(= \partial_t \mathfrak{p}_t|_{t=0})$ satisfies linear constraints if there exists a solution v to $(\Delta_{g_0} - 2)v = 0$ with singular terms prescribed by $\vec{A}$.

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Is 2 an eigenvalue of Δ_g^{Fr}?

- When $\vec{\beta} \in (0,1)^k$: the only spherical metrics with eigenvalue 2 are footballs (Bochner's technique / integration by parts)
- When at least one $\beta_i > 1$: the argument would not work any more
- Examples of metrics with $2 \in \mathrm{Spec}(\Delta_g^{\mathrm{Fr}})$: footballs, "heart", branched covers of the standard sphere
- Relation to reducible metrics [Xu–Z, 2019]
  - Metrics with reducible monodromy all satisfy $2 \in \mathrm{Spec}(\Delta_g^{\mathrm{Fr}})$
  - A spectral condition to characterize reducible monodromy
- Relation to harmonic maps [Karphukhin–Z, 2021]
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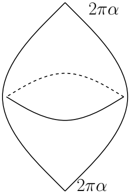
Two examples where $2 \in \mathrm{Spec}(\Delta_g^{\mathrm{Fr}})$



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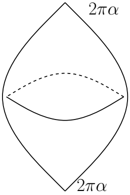

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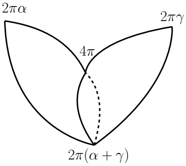
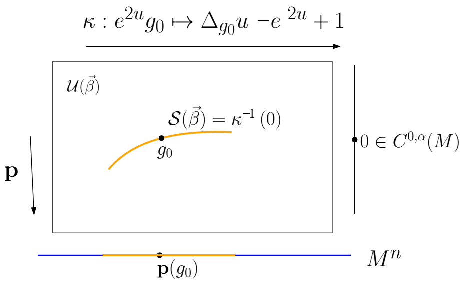


- The eigenfunctions on two footballs glue to a good eigenfunction ψ
- The complex gradient vector field of ψ again corresponds to conformal dilations
- This generates a family of spherical metrics with the same $\vec{\beta}$
- Rigidity: this family gives all spherical metrics with such $\vec{\beta}$ [Z '19]

A schematic picture



When 2 $\notin \mathrm{Spec}(\Delta^{\mathrm{Fr}}_{g_0})$,
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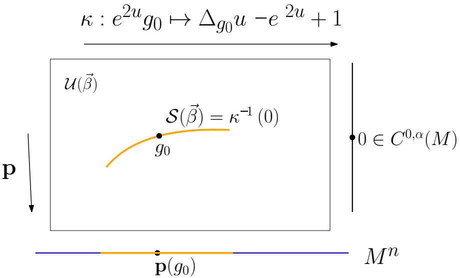

When 2 $\notin \mathrm{Spec}(\Delta_{g_0}^{\mathrm{Fr}})$,
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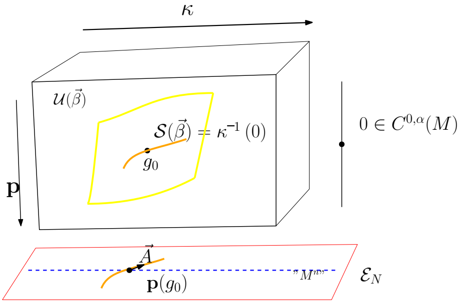

When 2 $\in \mathrm{Spec}(\Delta_{g_0}^{\mathrm{Fr}})$, in order to get a surjective map, we need to enlarge the parameter space to include splitting

A trichotomy theorem

Theorem (Mazzeo–Z, '19)

Let (M, g_0) be a spherical conic metric. Let $N = \sum_{j=1}^{k} \max\{[\beta_j], 1\}$. Let ℓ be the multiplicity of the eigenspace of $\Delta_{g_0}^{\mathrm{Fr}}$ with eigenvalue 2. There are three cases: $\ell = 0, 1 \leq \ell < 2N, \ell = 2N$.

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1. *(Local freeness) If* $\ell = 0$*, then* $g_0 \in \mathcal{S}(\vec{\beta})$ *has a smooth neighborhood parametrized by cone positions.*

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2. *(Partial rigidity) If* $1 \leq \ell < 2N$*, then there exists a* $2N - \ell$ *dimensional p-submanifold* $X \in \mathcal{E}_N$ *that parametrizes the cone position of nearby metrics.*

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3. *(Complete rigidity) If* $\ell = 2N$*, then there is no nearby spherical cone metric obtained by moving or splitting the cone points of* g_0*.*

Cone points collision

- To set up the nonlinear analysis, one needs to understand the splitting (or merging) of cone points
- We developed an $\mathcal{C}^\infty$ model that encodes information of such behaviors for all constant curvature conical metrics (not only spherical)
- Scale back the distance between two cone points (“blow up”)

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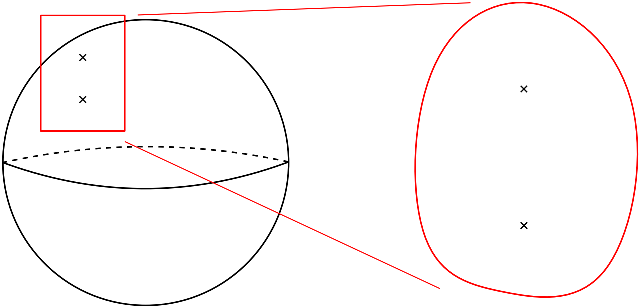
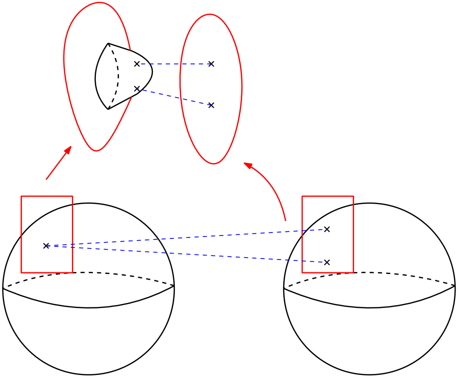


When two points collide

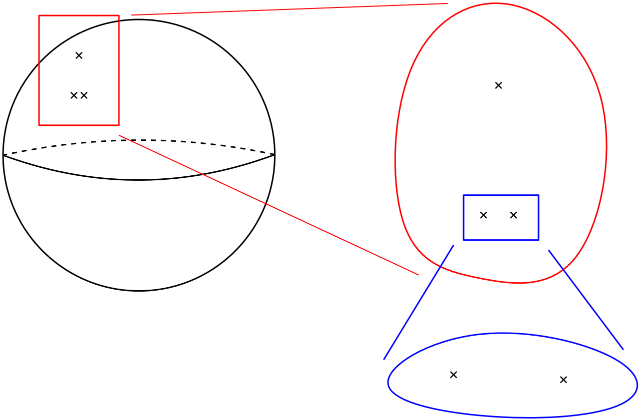
- Scale back the distance between two cone points (“blow up”)
- Half sphere at the collision point, with two cone points over the half sphere:



- Flat metric on the half sphere, and curvature K metric on the original surface

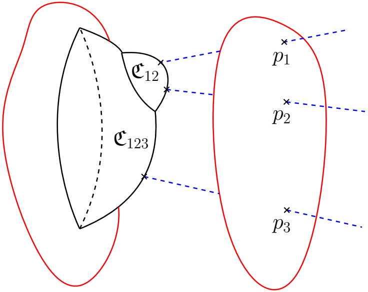
Iterative structure

- When there are several levels of distance: scale iteratively



Iterative structure

- "bubble over bubble" structure
- Higher codimensional faces from deeper scaling
- Flat conical metrics on all the new faces



- Iterative singular structures:
  Albin & Leichtnam & Mazzeo & Piazza '09-'19,
  Degeratu–Mazzeo '14, Kottke–Singer '15-'18,
  Albin–Gell-Redman '17, Albin–Dimakis–Melrose '19, ...

Results on fiber metric degenerations

Theorem (Mazzeo–Z, '17)

For any given $\vec{\beta}$, the family of constant curvature metrics with conical singularities is polyhomogeneous on this resolved space.*

- *The metric family can be hyperbolic / flat (with any cone angles), or spherical (with angles less than 2π, except footballs)
- Polyhomogeneity: existence of full asymptotic expansions, except the powers might include non-integers
- Solving the curvature equation uniformly using new coordinates

$$\Delta_{g_0} u - Ke^{2u} + K_{g_0} = 0$$

- The bubbles with flat conical metrics represent the asymptotic properties of merging cones
- We then applied this machinery to understand the big cone angle case [Mazzeo–Z '19]

Linear constraints given by eigenfunctions

- The splitting creates extra dimensions, which fills up the cokernel of the linearized operator $\Delta_g^{\mathrm{Fr}} - 2$
- The direction of admissible splitting is determined by the expansion of the eigenfunctions
- Each 2-eigenfunction gives a $2N$-tuple $\vec{b}$
- The tangent of splitting directions are given by vectors $\vec{A}$ that are orthogonal to all such $\vec{b}$ (linear constraints)
- The bigger dimension of eigenspace, the more constraint on the direction of splitting
- How to get the splitting direction from $\vec{A}$: “almost” factorizing polynomial equations using Bezout’s theorem

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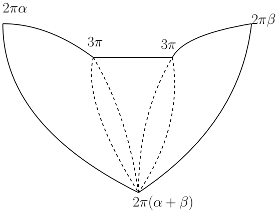
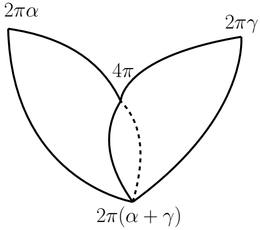
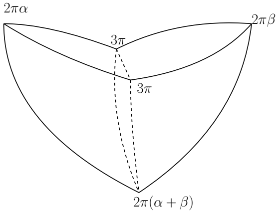
An example: open-heart surgery

- We obtain a deformation rigidity for the “heart”
- The cone point with angle 4π is split into two separate points
- In the equal splitting case: $4\pi \to (3\pi, 3\pi)$
- The spectral data dictates which splitting is possible:



Yes





No

Thank you for your attention!