Constant curvature conical metrics

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Joint works with Rafe Mazzeo, Bin Xu and Misha Karphukhin
Outline

1. Uniformization with conical singularities
2. Deformation rigidity
3. Compactified configuration space
Constant curvature metrics on Riemann surfaces

- Classical uniformization theorem: for a given Riemann surface, there is a unique (smooth) constant curvature metric

\[(\text{Gauss–Bonnet})\quad \chi(\Sigma) = \frac{1}{2\pi} KA\]
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- A constant curvature metric with conical singularities is a smooth metric with constant curvature, except near $p_j$ the metric is asymptotic to a cone with angle $2\pi \beta_j$

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\chi(\Sigma, \vec{\beta}) := \chi(\Sigma) + \sum_{j=1}^{k} (\beta_j - 1) = \frac{1}{2\pi} K A
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- Near a cone point with angle $2\pi \beta$, in geodesic polar coordinates

\[g = \begin{cases} 
  dr^2 + \beta^2 r^2 d\theta^2 & K = 0 \quad \text{(flat)} \\
  dr^2 + \beta^2 \sin^2 r d\theta^2 & K = 1 \quad \text{(spherical)} \\
  dr^2 + \beta^2 \sinh^2 r d\theta^2 & K = -1 \quad \text{(hyperbolic)} 
\end{cases}\]

- In conformal coordinates $z = (\beta r)^{1/\beta} e^{i\theta}$, \[g = f(z)|z|^{2(\beta - 1)}|dz|^2\]
Some examples of spherical conical metrics

Branched covers of constant curvature surfaces

"Heart": footballs glued along geodesics
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Spherical footballs

$4\pi$

$2\pi\alpha$

$2\pi(\alpha + \gamma)$
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The study of constant curvature conical metrics is related to:

- Magnetic vortices: solitons of gauged sigma-models on a Riemann surface
- Mean Field Equations: models of electro-magnetism
- Toda system: multi-dimensional version
- Higher dimensional analogue: Kähler–Einstein metrics with conical singularities
- Hyperbolic conical metrics: bridges between pointed and unpointed Riemann moduli spaces

This subject can be approached in many ways:

- PDE: singular Liouville equations
- Complex analysis: developing maps and Schwarzian derivatives
- Synthetic geometry: cut-and-glue
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Consider the following “conical data”:
- $n$ distinct points $p = (p_1, \ldots, p_n)$
- Angle data $\vec{\beta} = (\beta_1, \ldots, \beta_n)$, $\beta_i \in \mathbb{R}^+ \setminus \{1\}$
- Conformal structure $c$ given by the underlying Riemann surface

**Question**

Given conical data $(p, \vec{\beta}, c)$, does there exist a unique constant curvature conical metric with this data?
When uniformization holds

**Theorem (Heins ’62, McOwen ’88, Troyanov ’91, Luo–Tian ’92)**

For any compact Riemann surface \((M, c)\) and conical data \((p, \vec{\beta})\) with

- \(\chi(M, \vec{\beta}) \leq 0\); or
- \(\chi(M, \vec{\beta}) > 0, \ \vec{\beta} \in T \subset (0, 1)^k\)

there is a unique constant curvature conical metric with this data.

**Theorem (Mazzeo–Weiss ’15)**

If \(\vec{\beta} \in (0, 1)^k\), then there is a well-defined \((6\gamma - 6 + 3k)\)-dimensional moduli space.
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If \(\vec{\beta} \in (0, 1)^k\), then there is a well-defined \((6\gamma - 6 + 3k)\)-dimensional moduli space.
The remaining case: $\chi(M, \vec{\beta}) > 0$, at least one of the angles greater than $2\pi$

- Uniformization fails in this case
- **Existence**: constraints on conical data $(\rho, \vec{\beta}, c)$
  Mondello–Panov ’16, Chen–Lin ’17, Chen–Kuo–Lin–Wang ’18 ...
- **Uniqueness**: usually fails
  Chen–Wang–Wu–Xu ’14, Eremenko ’17, Bartolucci–De Marchis–Malchiodi ’11 ...
- **Deformation**: obstructions exist [Z ’19]

**Literature**:
- Troyanov ’91, Bartolucci & Carlotto & De Marchis & Malchiodi ’02–’19, Chen & Kuo & Lin & Wang ’02–’19, Umehara & Yamada ’00, Eremenko & Gabrielov & Tarasov ’01–’21, Xu ’14–’19, Mondello & Panov ’16–’21, Dey ’17
Spherical metrics with large cone angles

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Outline of the main result

Our results provide new understanding of the local structure of the moduli space where it is not smoothly parametrized:

**Theorem (Mazzeo–Z ’19)**

- The local deformation with respect to \((c, p, \vec{\beta})\) has rigidity precisely when \(2 \in \text{Spec}(\Delta^{\text{Fr}}_g)\);
- It can be “desingularized” by adding more coordinates via splitting of cone points.

Understanding this problem through a nonlinear PDE:

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\left\{ \text{Constant curvature } K \text{ conical metrics} \right\} \leftrightarrow \left\{ \text{Solutions to the Liouville equation} \right\}
\]

\[
\Delta_{g_0} u - Ke^{2u} + K_{g_0} = 0
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Here \(g_0\) is either a smooth metric (then \(u\) has singularities); or a conical metric with the given conical data (then \(u\) is bounded).
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Here \(g_0\) is either a smooth metric (then \(u\) has singularities); or a conical metric with the given conical data (then \(u\) is bounded).
Fix $\vec{\beta}$ for simplicity

- $\mathcal{U}(\vec{\beta})$: total space of conical metrics with angles $\vec{\beta}$
- $S(\vec{\beta})$: space of spherical conical metrics with angles $\vec{\beta}$
- $\kappa$: curvature map
- $p$: map to the cone positions
- The linearized operator is $\Delta g_0 - 2$
- When $2 \notin \text{Spec}(\Delta_{g_0}^{Fr})$, implicit function theorem applies to get a neighborhood of $p(g_0)$
- Otherwise we have a problem
Deformation and linear obstructions

- Fix a spherical conical metric $g_0 \in S(\vec{\beta})$. We study local deformations $g_t : (-\epsilon, \epsilon) \to S(\vec{\beta})$ and cone point positions $p_t = p(g_t)$.

- We have $g_t = e^{2u_t}g_0$ where $u_t$ solve the singular Liouville equation

$$\Delta g_0 u_t - e^{2u_t} + 1 = 0$$

Linearized equation: $(\Delta g_0 - 2)v = 0$ where $v := \partial_t u_t|_{t=0}$

- If $v \in \ker(\Delta_{Fr} g_0 - 2)$ where $\Delta_{Fr} g_0$ is the Friedrichs Laplacian, then $\partial_t p_t|_{t=0} = 0$: obstruction to injectivity of $p$.

- $\partial_t p_t|_{t=0}$ gives the singular terms of $v$ (those not in the Friedrichs domain). If $\ker(\Delta_{Fr} g_0 - 2) \neq 0$ then it might be impossible to find a solution with given singular terms: obstruction to surjectivity of $p$.

- We say $\vec{A}(= \partial_t p_t|_{t=0})$ satisfies linear constraints if there exists a solution $v$ to $(\Delta g_0 - 2)v = 0$ with singular terms prescribed by $\vec{A}$. 
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Is 2 an eigenvalue of $\Delta_{g}^{Fr}$?

- When $\vec{\beta} \in (0, 1)^k$: the only spherical metrics with eigenvalue 2 are footballs (Bochner’s technique / integration by parts)

- When at least one $\beta_i > 1$: the argument would not work any more

- Examples of metrics with $2 \in \text{Spec}(\Delta_{g}^{Fr})$: footballs, “heart”, branched covers of the standard sphere

- Relation to reducible metrics [Xu–Z, 2019]
  - Metrics with reducible monodromy all satisfy $2 \in \text{Spec}(\Delta_{g}^{Fr})$
  - A spectral condition to characterize reducible monodromy

- Relation to harmonic maps [Karphukhin–Z, 2021]
  - A harmonic map to $S^2$ produces such eigenfunctions
  - There is an algebraic description of the existence
  - Deformation of harmonic maps -> the dimension of the eigenspace can be arbitrarily large
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Two examples where $2 \in \text{Spec}(\Delta^\text{Fr}_g)$

- There is one eigenfunction $\Delta^\text{Fr}_g \phi = 2\phi$
- Take coordinate $z$ centered on the north pole, then the complex gradient vector field of $\phi$ is given by $-z\partial_z$, which corresponds to conformal dilations
Two examples where $2 \in \text{Spec} \left( \Delta_{g}^{\text{Fr}} \right)$

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- Take coordinate $z$ centered on the north pole, then the complex gradient vector field of $\phi$ is given by $-z \partial_{z}$, which corresponds to conformal dilations.

- The eigenfunctions on two footballs glue to a good eigenfunction $\psi$
- The complex gradient vector field of $\psi$ again corresponds to conformal dilations.
- This generates a family of spherical metrics with the same $\vec{\beta}$
- **Rigidity**: this family gives all spherical metrics with such $\vec{\beta}$ [Z ’19]
A schematic picture

When $2 \notin \text{Spec}(\Delta_{g_0}^\text{Fr})$, implicit function theorem applies to get a neighborhood of $p(g_0)$
A schematic picture

\[ \kappa : e^{2u}g_0 \mapsto \Delta g_0 u - e^{2u} + 1 \]

When \( 2 \notin \text{Spec}(\Delta^F_{g_0}) \), implicit function theorem applies to get a neighborhood of \( p(g_0) \).

When \( 2 \in \text{Spec}(\Delta^F_{g_0}) \), in order to get a surjective map, we need to enlarge the parameter space to include splitting.
Theorem (Mazzeo–Z, ’19)

Let \((M, g_0)\) be a spherical conic metric. Let \(N = \sum_{j=1}^{k} \max\{[\beta_j], 1\}\). Let \(\ell\) be the multiplicity of the eigenspace of \(\Delta_{g_0}^{\text{Fr}}\) with eigenvalue 2. There are three cases: \(\ell = 0, 1 \leq \ell < 2N, \ell = 2N\).
A trichotomy theorem

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1. (Local freeness) If \(\ell = 0\), then \(g_0 \in S(\vec{\beta})\) has a smooth neighborhood parametrized by cone positions.
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1. **(Local freeness)** If \(\ell = 0\), then \(g_0 \in S(\overrightarrow{\beta})\) has a smooth neighborhood parametrized by cone positions.

2. **(Partial rigidity)** If \(1 \leq \ell < 2N\), then there exists a \(2N - \ell\) dimensional \(p\)-submanifold \(X \in \mathcal{E}_N\) that parametrizes the cone position of nearby metrics.
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3. **(Complete rigidity)** If \(\ell = 2N\), then there is no nearby spherical cone metric obtained by moving or splitting the cone points of \(g_0\).
Cone points collision

- To set up the nonlinear analysis, one needs to understand the splitting (or merging) of cone points.
- We developed an $C^\infty$ model that encodes information of such behaviors for all constant curvature conical metrics (not only spherical).
- Scale back the distance between two cone points ("blow up")
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When two points collide

- Scale back the distance between two cone points ("blow up")
- Half sphere at the collision point, with two cone points over the half sphere:
  
  ![Diagram showing the collision of two points and the resulting half sphere.]

- Flat metric on the half sphere, and curvature $K$ metric on the original surface
Iterative structure

- When there are several levels of distance: scale iteratively
Iterative structure

- “bubble over bubble” structure
- Higher codimensional faces from deeper scaling
- Flat conical metrics on all the new faces

Iterative singular structures:
Albin & Leichtnam & Mazzeo & Piazza ’09–’19,
Degeratu–Mazzeo ’14, Kottke–Singer ’15–’18,
Albin–Gell-Redman ’17, Albin–Dimakis–Melrose ’19, ...........
Results on fiber metric degenerations

Theorem (Mazzeo–Z, ’17)

For any* given $\vec{\beta}$, the family of constant curvature metrics with conical singularities is polyhomogeneous on this resolved space.

- *The metric family can be hyperbolic / flat (with any cone angles), or spherical (with angles less than $2\pi$, except footballs)
- Polyhomogeneity: existence of full asymptotic expansions, except the powers might include non-integers
- Solving the curvature equation uniformly using new coordinates

\[ \Delta g_0 u - Ke^{2u} + K_{g_0} = 0 \]

- The bubbles with flat conical metrics represent the asymptotic properties of merging cones
- We then applied this machinery to understand the big cone angle case [Mazzeo–Z ’19]
Linear constraints given by eigenfunctions

- The splitting creates extra dimensions, which fills up the cokernel of the linearized operator $\Delta^\text{Fr}_g - 2$
- The direction of admissible splitting is determined by the expansion of the eigenfunctions
- Each 2-eigenfunction gives a $2N$-tuple $\vec{b}$
- The tangent of splitting directions are given by vectors $\vec{A}$ that are orthogonal to all such $\vec{b}$ (linear constraints)
- The bigger dimension of eigenspace, the more constraint on the direction of splitting
- How to get the splitting direction from $\vec{A}$: “almost” factorizing polynomial equations using Bezout’s theorem
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An example: open-heart surgery

- We obtain a deformation rigidity for the “heart”
- The cone point with angle $4\pi$ is split into two separate points
- In the equal splitting case: $4\pi \rightarrow (3\pi, 3\pi)$
- The spectral data dictates which splitting is possible:
Thank you for your attention!