Equivariant K-stability and valuative criteria

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Partially based on joint work with Yuchen Liu

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- The dimension of X is denoted by n.

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- Originally, K-stability is defined via degenerations (test configurations), and in general, it is not easy to verify.
- Equivariant K-stability is defined via equivariant test configurations.
- For $\mathbb{Q}$-Fano varieties with large symmetry, checking equivariant K-stability is sometimes easier. (E.g. $\mathbb{P}^n$, V_{22}^*)

Recent progress

Conjecture

Let X be a $\mathbb{Q}$-Fano variety and G a reductive group action on X. If X is G-equivariantly K-semistable (resp. G-equivariantly K-polystable), then X is K-semistable (resp. K-polystable).

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Equivariant test configuration

A **test configuration** $(\mathcal{X}, \mathcal{L})$ of $(X, -rK_X)$ is a $\mathbb{G}_m$-equivariant degeneration of X over $\mathbb{A}^1$:

$$\begin{array}{ccccc} X \times \{1\} & \hookrightarrow & X \times \mathbb{C}^* & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \{1\} & \hookrightarrow & \mathbb{C}^* & \hookrightarrow & \mathbb{A}^1. \end{array}$$

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G-action on X induces a G-action on $X \times \mathbb{C}^*$:

$$g(x, t) = (\lambda_t \circ g \circ \lambda_t^{-1}(x), t), \ \ g \in G.$$

where λ_t is the $\mathbb{G}_m$-action on the test configuration.

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where λ_t is the $\mathbb{G}_m$-action on the test configuration.
The test configuration is called G-**equivariant** if the above G-action extends to an action on $(\mathcal{X}, \mathcal{L})$ commuting with the $\mathbb{G}_m$-action.

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The **generalized Futaki invariant** of a normal $(\mathcal{X}, \mathcal{L})$ can be computed by

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \frac{1}{(n+1)(-K_X)^n} \left(\frac{n}{r^{n+1}} \overline{\mathcal{L}}^{n+1} + \frac{n+1}{r^n} \overline{\mathcal{L}}^n \cdot K_{\overline{\mathcal{X}}/\mathbb{P}^1} \right)$$

Norm of a test configuration

Let

$$\begin{array}{ccc} & \mathcal{Y} & \\ {}^{p}\swarrow & & \searrow^{q} \\ X \times \mathbb{P}^1 & \dashrightarrow & \overline{\mathcal{X}}. \end{array}$$

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The **Non-Archimedean norm** of $(\mathcal{X}, \mathcal{L})$ can be computed by

$$J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := \frac{p^*(-K_{X\times\mathbb{P}^1/\mathbb{P}^1})^n \cdot q^*\bar{\mathcal{L}}}{(-K_X)^n} - \frac{\bar{\mathcal{L}}^{n+1}}{(n+1)(-rK_X)^n}.$$

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$J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ is one of the several equivalent norms of test configurations that can be used to define uniform K-stability.

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Note: If we take $G = \{e\}$, then we recover the usual K-stability notions.

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  $\mathbb{P}^n$ is uniformly G-equivariantly K-stable, since there is no nontrivial G-equivariant test configuration for $\mathbb{P}^n$.
- Therefore the equivariant K-stability conjecture is not true for uniform K-stability.

Examples (Fano threefolds of degree 22)

Example (Cheltsov-Shramov)

V_{22}^* denotes a class of Fano threefolds of degree 22 admitting G-action with $G = \mathbb{C}^* \rtimes \mathbb{Z}/2$. All but two of them have G-equivariant alpha invariants to be $4/5$.

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According to the equivariant Tian's criterion, those $X \in V_{22}^*$ with $\alpha_G(X) = 4/5$ are uniformly G-equivariantly K-stable and hence K-polystable (Kähler-Einstein) due to the equivariant K-stability conjecture.

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- $k = n$: X is K-stable if B is smooth. (R. Dervan)
- $k < n$: B is Fano, K-(semi/poly)stability of $X \Leftrightarrow$ K-(semi/poly)stability of B. (Liu-Z)

Valuative criterion

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where

$$\tau(E) = \sup\{t | \operatorname{vol}_Y \left(\pi^*(-K_X) - tE\right) > 0\}$$

is the pseudo-effective threshold of E.

Valuative criterion (Cont'd)

The following two invariants are involved in the valuative criterion:

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Theorem (Fujita, Li)

A Fano variety X is

1. *K-semistable if and only if $\beta(E) \geq 0$ for every prime divisor E over X;*
2. *uniformly K-stable if and only if there exists $\delta \in (0, 1)$, such that $\beta(E) \geq \delta j(E)$ for every prime divisor E over X.*

Pseudovaluations

Let $G < \mathrm{Aut}(X)$ be a group action on X. For any valuation v on X, we define

$$G \cdot v := \inf_{g \in G} g \cdot v,$$

where $g \cdot v$ is the valuation given by $g \cdot v(f) = v(f \circ g)$ for any $f \in \mathbb{C}(X)$.

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For any nonnegative real number x, the ideal sheaf

$$\mathfrak{a}_x(G \cdot v) = \bigcap_{g \in G} \mathfrak{a}_x(g \cdot v)$$

collects regular functions of vanishing order at least x with respect to all $g \cdot v$'s.

Equivariant valuative criterion

Let E be a divisor over X. Define

$$S_X^G(E) := \frac{1}{(-K_X)^n} \int_0^{\tau^G(E)} \mathrm{vol}_X\left(\mathcal{O}_X(-K_X) \otimes \mathfrak{a}_x(G \cdot \mathrm{ord}_E)\right)\, dx,$$

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The equivariant invariants we need for the equivariant valuative criterion are

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Note that the above invariants coincide with the usual ones if E is G-invariant.

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where R runs through all ramification divisors and e_R is the ramification index of R. For each branch divisor B_i on Y, denote by e_i the ramification index of R with $\sigma(R) = B_i$.
Then we have

$$K_X = \sigma^*(K_Y + B),$$

where $B = \sum_i \left(1 - \frac{1}{e_i}\right) B_i$.

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Idea of the proof:

- "Only if" part: comparing beta invariants of corresponding prime divisors;
- "If" part: comparing generalized Futaki invariants of corresponding test configurations.

Main Result

Theorem (Liu-Z)

Let X be a $\mathbb{Q}$-Fano variety and G a finite group action on X. If X is G-equivariantly K-semistable (resp. G-equivariantly K-polystable), then X is K-semistable (resp. K-polystable).

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X is G-equivariantly K-semistable $\Rightarrow$ (Y, B) is K-semistable $\xRightarrow{\text{Blum-Liu}}$ $(Y, B + \epsilon D')$ is uniformly K-stable, for some $D' \in |-(K_Y + B)|_{\mathbb{Q}}$

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$\Rightarrow (X, \epsilon D)$ is Kähler-Einstein, where $D = \sigma^* D' \in |-K_X|_{\mathbb{Q}}$

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X is G-equivariantly K-semistable $\Rightarrow (Y, B)$ is K-semistable $\xrightarrow{\text{Blum-Liu}} (Y, B + \epsilon D')$ is uniformly K-stable, for some $D' \in |-(K_Y + B)|_{\mathbb{Q}}$ $\xrightarrow{\text{Uniform YTD (Li-Tian-Wang)}} (Y, B + \epsilon D')$ is Kähler-Einstein $\Rightarrow (X, \epsilon D)$ is Kähler-Einstein, where $D = \sigma^* D' \in |-K_X|_{\mathbb{Q}}$ $\xrightarrow{\text{Berman}} (X, \epsilon D)$ is K-semistable $\xrightarrow{\text{Blum-Liu}} X$ is K-semistable.

Main Result

Theorem (Liu-Z)

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Note: The proof of the uniform YTD conjecture in L-T-W is analytic.

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Assume X is strictly K-semistable, then there is a special test configuration $\mathcal{X}$ degenerating X to some X_0 which is K-polystable.

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A test configuration $\mathcal{X}$ degenerating X to X_0 corresponds to a morphism $\phi : [\mathbb{A}^1/\mathbb{G}_m] \to [\operatorname{Spec} A/H]$ with $\phi(1) = [X]$ and $\phi(0) = [X_0]$.

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Equivalently, we have a 1-parameter subgroup: $\lambda : \mathbb{G}_m \to H$ with

$$\lim_{t \to 0} \lambda(t) \cdot [X] = [X_0].$$

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This gives us the desired G-equivariant test configuration $\mathcal{X}'$.
$\Rightarrow$ Contradiction, and hence X is K-polystable.

Thank you!