4-dimensional aspects of Ricci flow

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MIT

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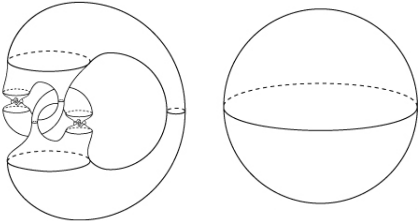
Collaborations with **A. Deruelle** and **K. Naff**.

Geometry and Topology Seminar,
CIRGET – UQAM.

“Best” geometry

Given a **topology**, M^n, is there a *best* **geometry**, (M^n, g) ?



“Better” geometry → easier to understand the topology.

Oversimplifying :

- **Best** geometries are Einstein, and
- Ricci flow **optimizes** geometries.

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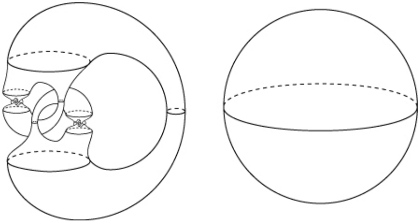

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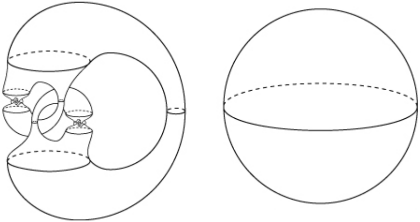

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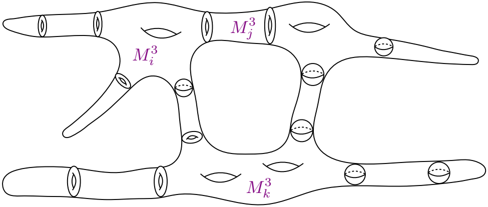
Hamilton, Perelman 2003

Any 3-manifolds M^3 decomposes in M_i^3 admitting "best" **model geometries**,



For $n = 3$, **Ricci flow** provides a **decomposition** $M^3 = \bigcup_i M_i^3$:

1. **Finite-time singularities** : separation along $\mathbb{S}^2$ (and quotient).
2. **Long-term behavior** : separation along $\mathbb{T}^2$ (and quotient) of *thick* **hyperbolic** regions and *thin* regions.

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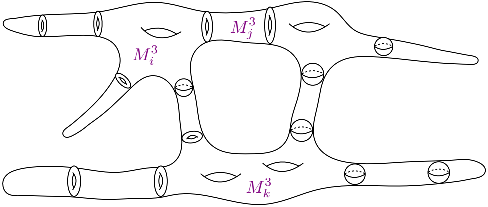

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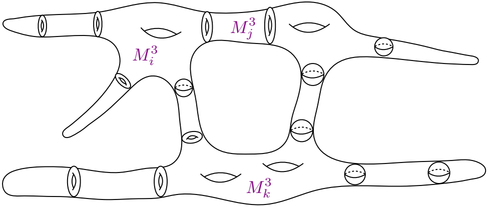

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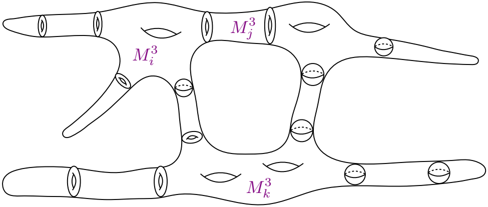

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A 4-dimensional geometrization ?

Yau asks in 2000 : can any M^4 (simply connected, orientable...) be decomposed into M_i^4

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Einstein metrics in dimension 4

An **Einstein** metric satisfies

$$\exists\Lambda \in \mathbb{R}, \ \ \mathrm{Ric}(g) \ = \ \Lambda g. \quad \text{“}\mathrm{Ric}(g) \approx -\frac{1}{2}\Delta g\text{”}.$$

In dimension 4, Einstein metrics minimize complexities.

$$g \ \mapsto \ \underbrace{\int_{M^4} |\,\mathrm{Rm}_g\,|^2 dv_g}_{\textit{complexity}} \ \geqslant \ \underbrace{8\pi^2\chi(M^4)}_{\textit{topological}}\ ; \text{ equality } \iff \ g \text{ is Einstein.}$$

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Note : Einstein metrics are not “manageable” from dimension 5 : they form complicated singularities.

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Spectacular applications in **geometry** and **topology**, for instance :

- **Dimension** 3 **:** Hamilton'82, Perelman'02, Bamler-Kleiner'20...
- **Any dimension** n **:** Böhm-Wilking '08, Brendle-Schoen '09, Bamler'20...

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Main questions on Ricci flow

Question : What can we say about Ricci flow **in dimension** 4 without **strong** (i.e. $\mathrm{Rm} > 0$, PIC...) assumptions ?

The most interesting dimension

(According to 4-dimensional topologists and geometers)

Why dimension 4 ?

1. It is the next dimension after 3.
2. Ricci flows are not “manageable” from dimension 5.
3. Dimension 4 is where most open questions are in topology.
4. **Specific techniques/theories** in dimension 4 : Selfduality, special properties of Einstein metrics, Yang-Mills connections, Penrose twistor methods, (hyper)Kähler geometry, Seiberg-Witten theory, index theorems...

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A distant goal

Question (Bamler'20, Tian,...)

Is Ricci flow a reasonable approach to the 11/8-conjecture ?

Would complete the classification of smooth simply connected 4-manifolds up to **homeomorphism**.

11/8 conjecture and why Ricci flow

Conjecture (Matsumoto'82)

Let M be a 4-dimensional oriented, closed, simply connected, spin manifold. Then :

$$b_2(M) \geqslant \frac{11}{8}|\tau(M)|,$$

where b_2 is the second Betti number and τ the signature.

Note : there is equality on $K3$. The inequality with $\frac{10}{8}$ is known.

Intuition/Strategy (Bamler)

$$b_2(M) \geqslant \frac{11}{8}|\tau(M)| \text{ ???}$$

- *Finite time singularities :* have **positive scalar curvature** so $\tau(M) = 0$, and
- *Infinite time :* Einstein pieces satisfy the better **Hitchin Thorpe inequality** :

$$(2 + b_2(M)) \geqslant \frac{12}{8}|\tau(M)|.$$

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1. Einstein 4-manifolds and their degenerations
2. Stability of orbifold singularities (with A. Deruelle)
3. Selfduality in dimension 4 (with K. Naff)
4. Conclusion and perspectives

Compactness theorem for Einstein 4-manifolds

The Gromov-Hausdorff (**GH**) distance is the most **general** for Einstein metrics and their degenerations.

Theorem (Anderson, Bando-Kasue-Nakajima'89, ...'70's-80's)

A sequence $g_i \in \mathbf{E}(M^4)$ with bounded diameter always ***GH****-subconverges to an* ***Einstein*** *4-****orbifold****.*

The worst singular limits are not that bad in dimension 4.

Issue : this GH-convergence is too weak for applications.

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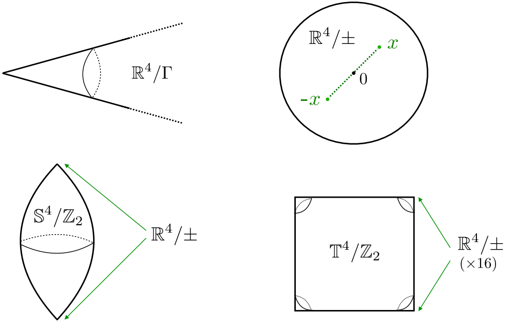
An **Einstein orbifold** (with isolated singularities) :

- is **smooth** and Einstein away from a **finite** number of points
- with $\mathbb{R}^4/\Gamma$-singularities, for **finite** $\Gamma \subset SO(4)$ fixing only 0.



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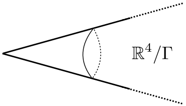

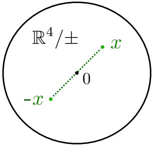

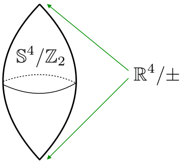

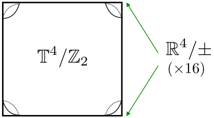

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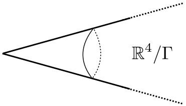

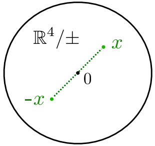

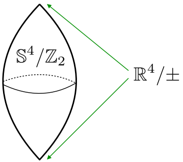

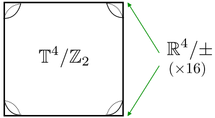

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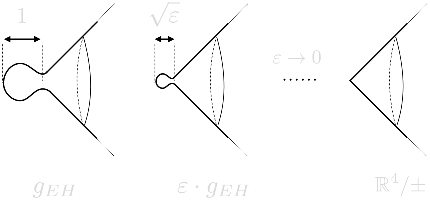
Bubbling off of Ricci-flat ALE manifolds

Orbifold singularity formation

A Ricci-flat **Asymptotically Locally Euclidean (RFALE)**

- satisfies $\mathrm{Ric} \equiv 0$ and
- is asymptotic to $\mathbb{R}^4/\Gamma$ for $\Gamma \subset SO(4)$ fixing only 0.

Example : The Eguchi-Hanson'79 (EH) metric, g_{EH}, is Ricci-flat, asymptotic to $\mathbb{R}^4/\pm$.



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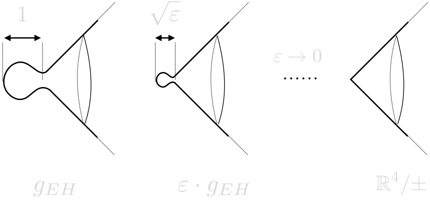

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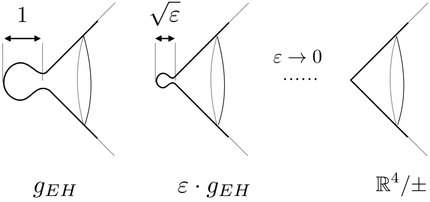
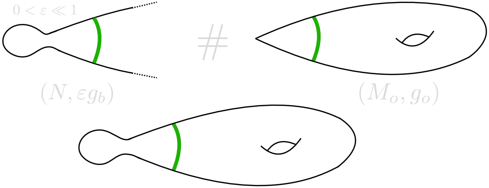


Gluing-perturbation

- $(N^4, g_b) \underset{\infty}{\sim} \mathbb{R}^4/\Gamma$ with $\mathrm{Ric}(g_b) \equiv 0$, and
- $(M_o^4, g_o) \underset{p}{\sim} \mathbb{R}^4/\Gamma$ with $\mathrm{Ric}(g_o) = \Lambda g_o$.

We define the **naïve desingularization** (M, g^D_ε).



$(M, g^D_\varepsilon) := (N \# M_o, \varepsilon g_b \# g_o)$

Question : Perturbation to Einstein ?

Gluing-perturbation

- $(N^4, g_b) \underset{\infty}{\sim} \mathbb{R}^4/\Gamma$ with $\mathrm{Ric}(g_b) \equiv 0$, and
- $(M^4_o, g_o) \underset{p}{\sim} \mathbb{R}^4/\Gamma$ with $\mathrm{Ric}(g_o) = \Lambda g_o$.

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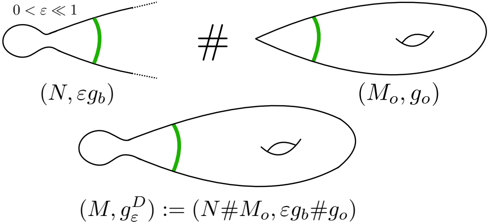
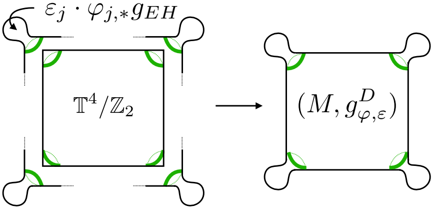


Question : Perturbation to Einstein ?

Kummer construction on K3

Page'78, Gibbons-Pope'79, Topiwala'87, Lebrun-Singer'94, Donaldson'12, O.'19

EH metrics glued at scales $\varepsilon_j > 0$ in the **same orientation** ($\varphi_j \in SO(4)$) to $\mathbb{T}^4/\mathbb{Z}_2$.



Can be perturbed to a Ricci-flat metric.

Perturbation step

There was a gap :

- **In the 80's :** Study of the **general** GH-degenerations.
- **Since the 80's :** Construction of **specific** $C^{2,\alpha}_{\beta}$-degenerations.

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If Einstein : GH-degenerations $\iff$ $\tilde{C}^{2,\alpha}_{\beta}$-degenerations.

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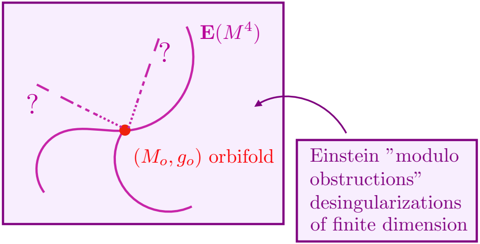
A Kuranishi map around singular metrics

Biquard'13,'16, O.'19 : Einstein modulo obstructions.

$$\mathrm{Ric}(\hat{g}_\varepsilon) - \Lambda\hat{g}_\varepsilon = \hat{\mathbf{o}}_\varepsilon \in \{\text{Obstructions}\} \approx \text{"coker"}(\text{linearization}).$$

O.'19 : $\mathbf{E}(M^4) = \{\text{Einstein geometries on } M^4\}$,



Einstein "modulo obstructions" desingularizations of finite dimension

General obstruction : A classical question

Converse (Folklore, 90's, Naber) : Is **any** Einstein 4-orbifold GH-limit of Einstein 4-manifolds ?

Theorem (O.'21 , "No !")

$\mathbb{S}^4/\mathbb{Z}_2$ ***cannot*** *be GH-limit of smooth Einstein 4-manifolds.*

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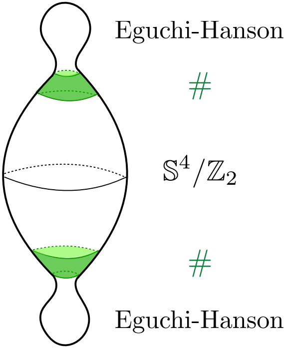
Common thread

Example with **expected** topology $M = \mathbb{S}^2 \times \mathbb{S}^2$.



$\mathbb{S}^4/\mathbb{Z}_2$ **cannot** be GH-limit of smooth Einstein metrics.

O.'21 : true for **any** RFALE (not classified), not just EH.

1. Einstein 4-manifolds and their degenerations
2. Stability of orbifold singularities (with A. Deruelle)
3. Selfduality in dimension 4 (with K. Naff)
4. Conclusion and perspectives

Degeneration of Ricci flows

in dimension 4

Orbifold degeneration of Ricci flows : Bamler-Zhang'15, Simon'15 and Bamler'16,'18, Appleton'19, Brendle-Kapouleas'17.

Theorem (Bamler'20, Compactness theorem)

Along a 4-dimensional Ricci flow, in a weak "$\mathbb{F}$"-sense

1. finite-time singularities : ***orbifold Ricci soliton*** $\mathrm{Ric}(g) + \mathrm{Hess}\, f = \Lambda g$, $\Lambda > 0$ *(e.g. $\mathbb{S}^4/\mathbb{Z}_2$).*
2. long term behavior : *thin parts and thick* ***Einstein orbifolds*** *(e.g. hyperbolic orbifolds).*

Issue : Bamler's notion of $\mathbb{F}$-convergence is too weak for applications.

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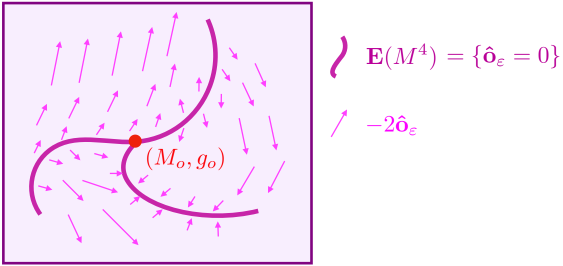
A center manifold analysis

O.'21, Deruelle-O.'23 : $\partial_t g = -2(\mathrm{Ric}(g) - \Lambda g) \approx -2\hat{\mathbf{o}}_\varepsilon$.



The obstructions $\hat{\mathbf{o}}_\varepsilon$ determine the **dynamic** of Ricci flow.

Einstein modulo obstructions and stability

O.'19'21 : in the simplest situation, $\|\hat{g}_\varepsilon - g^D_\varepsilon\|_{\tilde{C}^{2,\alpha}_\beta} \ll 1$,

$$\mathrm{Ric}(\hat{g}_\varepsilon) - \Lambda \hat{g}_\varepsilon \approx \gamma \cdot \varepsilon\partial_\varepsilon \hat{\boldsymbol{g}}_\varepsilon = \hat{\mathbf{o}}_\varepsilon \quad \in \quad \{\text{Obstructions}\}, \quad \gamma \in \mathbb{R}.$$

At leading order, $\partial_t g = -2(\mathrm{Ric}(g) - \Lambda g)$ only modifies the geometry in the bubble region.The scale ε evolves by

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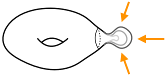
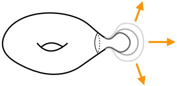


- **Stability :** if $\gamma > 0$, it **shrinks** and flows **back** to the orbifold. Caused by **negative** curvature.



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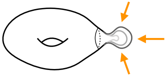

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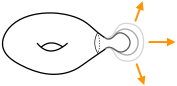

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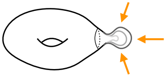

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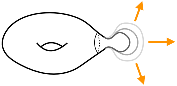

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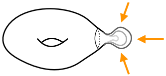

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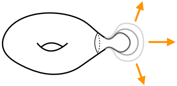

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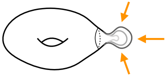

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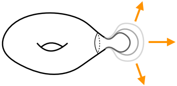

Case of Kronheimer's instantons

On Eguchi-Hanson and other Kronheimer's instantons, Kronheimer's period map parameters $\zeta_j \in \mathbb{R}^3 \approx \Lambda^+$ evolves by

$$\zeta_j' \approx \mathbf{R}^+(\zeta_j),$$

$\mathbf{R}^+ : \Lambda^+ \to \Lambda^+$ selfdual curvature of the orbifold at its singular point (3×3 matrix).

(In)stability of Einstein orbifolds

along Ricci flow

Definition (Linear stability (Deruelle-O. Upcoming))

An Einstein orbifold is **"linearly" stable** "at the singularity" p along Ricci flow if :

$$\mathbf{R}(p) < 0,\ \mathbf{R}(p) \in \Lambda^2_p \otimes \Lambda^2_p \text{ Riemannian curvature.}$$

Note : with **expected topology** (i.e. gluing Kronheimer's instantons), the condition is $\mathbf{R}^+(p) < 0$.

(In)stability of orbifold Ricci solitons

along Ricci flow

Deruelle-O. (upcoming) : The (in)stability conditions on **Ricci solitons** are the same if one replaces $\mathbf{R}$ by $\overline{\mathbf{R}}$ of Cao-Tran'16.

Note : on the way, obstruction to desingularizing Ricci solitons.

If $\mathrm{Ric} + \mathrm{Hess}_f = \Lambda g$ on the soliton, then :

$$\overline{\mathbf{R}}^+ = \mathbf{R}^+ + \frac{\Delta f}{4} I_{\Lambda^+} = \mathbf{R}^+ + \Big(\Lambda - \frac{R}{4}\Big) I_{\Lambda^+}.$$

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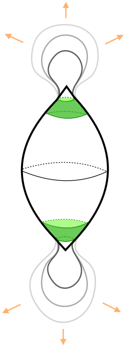
Dynamical instability



Deruelle-O.'23 : There is an **ancient** (renormalized) Ricci flow defined for $t \in (-\infty, 0]$:

$$\partial_t g_{\varepsilon(t)} = -2(\mathrm{Ric}(g_{\varepsilon(t)}) - 3g_{\varepsilon(t)})$$

with limit $\mathbb{S}^4/\mathbb{Z}_2$ at $-\infty$ and $\varepsilon'(t) \approx 1 \cdot \varepsilon(t)$

$$\boxed{\text{Ricci flow desingularizes } \mathbb{S}^4/\mathbb{Z}_2 \text{ – but not by Einstein metrics.}}$$

Particularities of this ancient flow

This ancient renormalized Ricci flow

- has **pinched positive Ricci curvature** (impossible in dimension 3 by Brendle-Huisken-Sinestrari'11), and
- has bounded selfdual curvature and traceless Ricci curvature.

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Idea of proof

We perturb the "almost Ricci flow" $\hat{g}_{\varepsilon(t)}$ with $\varepsilon' = -2\gamma \cdot \varepsilon$.
Requires an analysis of **parabolic** equations in *weighted Hölder spaces* extending ideas of Brendle-Kapouleas'17.

Let $L = -\nabla^*\nabla + \mathcal{R}$ be an *elliptic* operator on 2-tensors.

Key linear idea : let $\psi = \psi(t, x)$ be a time-dependent 2-tensor decaying at $t \to -\infty$, then, there exist unique $h = h(t, x) \perp \ker L$ and $\mathbf{o}_t, \mathbf{o}'_t \in \ker L$ decaying at $t \to -\infty$,

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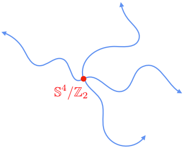
Conjecture (Deruelle-O.'23) : **stability** of orbifold singularities is the **core mechanism** for orbifold singularity formation.

→ $\mathbb{S}^4/\mathbb{Z}_2$ is "**unstable**" in **all** directions : it should **never** appear.



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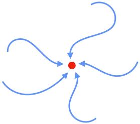

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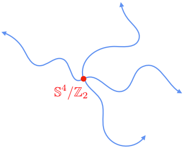

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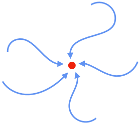

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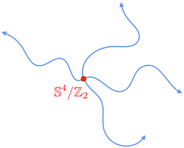

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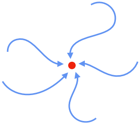

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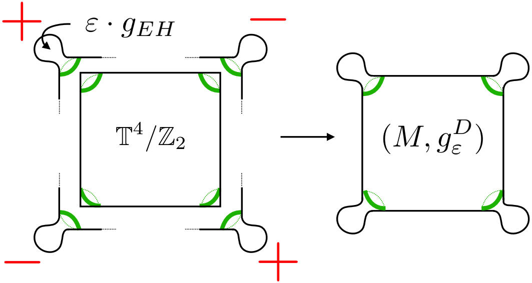

There is an **ancient Ricci flow** with limit $\mathbb{T}^4/\mathbb{Z}_2$ and EH growing (explained as "$\mathbf{R} > 0$" in O.'20).

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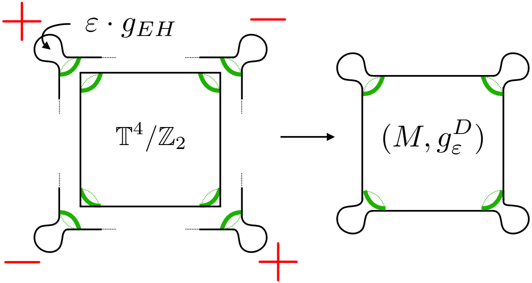

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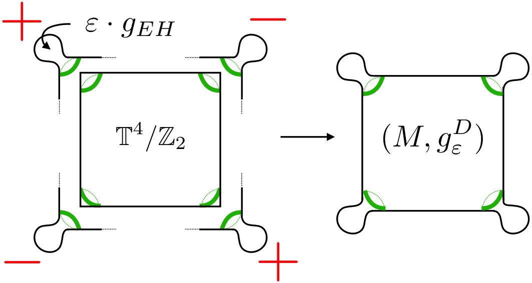

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A stability conjecture

Cohomogeneity 1, $U(2)$-invariant Ricci flow on $T^*\mathbb{S}^2$:

$$\text{EH} \quad \#_{\mathbb{RP}^3} \quad \text{Bryant}/\mathbb{Z}_2 \quad \#_{\mathbb{RP}^3} \quad \mathbb{RP}^3\times\mathbb{R}.$$

Close to the singular time :

$$\text{scale}(\text{EH}) \;\ll\; \text{scale}(\text{Bryant}/\mathbb{Z}_2) \;\ll\; \text{scale}(\mathbb{RP}^3\times\mathbb{R}).$$

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Stable and unstable orbifold Ricci solitons

Stable	Unstable
Hyperbolic	Spherical
Selfdual Einstein and *Scal* < 0	Selfdual Einstein and *Scal* > 0
Gaussian expander	Gaussian shrinker
Bryant's steady soliton	FIK compact shrinking solitons
Deruelle's expanders	Einstein metrics with *Scal* $\geqslant 0$(*)
	$\mathbb{T}^4/\mathbb{Z}_2$(**)

(*) Except maybe if $\mathbf{R}^+ = 0$.
(**) Except if on K3.

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A group theoretic “anomaly”

Many specificities of dimension 4 come from : **SO(4) is not simple.** All other SO(n) are simple.

In particular at the Lie algebra level,

$$\underbrace{\Lambda^2(\mathbb{R}^4)}_{\text{dim. }6} \approx \underbrace{\Lambda^2(\mathbb{R}^3)}_{\text{dim. }3} \oplus \underbrace{\Lambda^2(\mathbb{R}^3)}_{\text{dim. }3}.$$

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$\pm$-selfdual 2-forms

On (M^4, g), one defines $\Lambda^\pm_g = \ker(*_g \mp \mathrm{Id}_{\Lambda^2}) \subset \Lambda^2$, to find

$$\Lambda^2 = \Lambda^+_g \oplus \Lambda^-_g.$$

The Riemannian curvature $\mathbf{R} \in \Lambda^2 \otimes \Lambda^2$ decomposes as :

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where $W^\pm$ are the $\pm$-selfdual parts of the Weyl tensor W.

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PIC$_+$ Ricci flows

Question (Hamilton,...) : Can we develop a theory of Ricci flows with $\mathbf{R}^+ > 0$? Can we classify shrinking solitons $\mathbf{R}^+ > 0$?

Hamilton '86 :

$$\partial_t \mathbf{R}^+ = -\nabla^*\nabla \mathbf{R}^+ + (\mathbf{R}^+)^2 + \mathring{\mathrm{Ric}}^T \cdot \mathring{\mathrm{Ric}} + (\mathbf{R}^+)^\sharp, \text{ and}$$
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The bundle Λ^+_g and its curvature

Key point : Λ^+_g determines the **conformal class** $[g]$.

The Levi-Civita connection of Λ^+_g is denoted $T_g \in \Lambda^1 \otimes \Lambda^+_g$.

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$$F_g = F_g^+ + F_g^- \in \Lambda^2 \otimes \Lambda^+_g = (\Lambda^+_g \oplus \Lambda^-_g) \otimes \Lambda^+_g.$$

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The curvature of Λ^+_g never sees W^-, the $\mathbf{R}^-$ does not appear :

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→ Explains the formation of orbifold singularities.

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Index theorems and Ricci flow

Thanks to **Atiyah-Singer index theorem** in dimension 4 : $\tau(M) = \mathrm{index}(D_g)$ and $\chi(M) = b_2(M) - 2 = \mathrm{index}(d + d_g^*)$ (if simply connected).

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A **weighted Riemannian manifold** is a metric measure space $(M, g, e^{-f} dv_g)$ where $f : M \to \mathbb{R}$.

- Perelman's breakthrough relies on **weighted scalar curvature** $\mathrm{Scal}_f := \mathrm{Scal} + 2\Delta f - |\nabla f|^2$ to define his functionals :

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Towards the 11/8 conjecture ?

Main challenge : estimating $\mathrm{index}(D) = \mathrm{index}(D_f)$ and $\mathrm{index}(d + d^*) = \mathrm{index}(d_f + d_f^*)$ at **finite-time** singularities and for the **long term behavior** of Ricci flow.

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Thank you for your attention !