APPLICATIONS OF CHARACTER VARIETY METHODS TO DEHN SURGERY

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ABSTRACT. These are an expanded version of the lecture notes of a minicourse on the use of $SL(2,\mathbb{C})$-character varieties in Dehn surgery theory given as part of the ICERM Topical Workshop *Perspectives on Dehn Surgery*, held June 15–19, 2019 at ICERM in Providence, Rhode Island.

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Steven Boyer was partially supported by NSERC grant RGPIN 9446-2013.
2000 MATHEMATICS SUBJECT CLASSIFICATION. PRIMARY 57M25, 57M50, 57M99

1. Introduction

The 1987 publication of the paper *Dehn surgery on knots* by Marc Culler, Cameron McA. Gordon, John Luecke, and Peter Shalen was a milestone in 3-dimensional surgery theory ([CGLS]). Here is its main result:

The Cyclic Surgery Theorem. *Let M be a non-Seifert fibred knot manifold and α, β slopes on ∂M such that $\pi_1(M(\alpha))$ and $\pi_1(M(\beta))$ are cyclic. Then the distance between α and β is at most 1. Consequently there are at most three Dehn fillings of M with cyclic fundamental groups.*

Beyond the importance of this theorem, the paper set the model for framing exceptional surgery problems and provided powerful techniques for analysing them. The first half of the paper, due to Culler and Shalen, introduced representation-theoretic methods to deal with the case that M is hyperbolic. The second, due to Gordon and Luecke, developed graph-theoretic methods which could handle the non-hyperbolic case. But this is an overly simplistic assessment of the roles each half plays. In fact, they intertwine in subtle ways: When M is hyperbolic, there are situations when the graph theoretic methods of Gordon and Luecke are needed and reciprocally, there are situations when the results of the second half need to be supplemented by the methods of Culler and Shalen.

The goal of these notes is to provide an introduction to the $SL(2,\mathbb{C})$-character variety methods used in the proof of the cyclic surgery theorem and outline some of their applications to surgery theory. (See the chapter of this volume written by Gordon for an exposition of the graph theoretic methods in a related setting.) Perhaps surprisingly, our departure point in §2 is the relationship between essential surfaces in 3-manifolds and non-trivial actions of their fundamental groups on simplicial trees. The connection this has with representation theory is the focus of §3, where we explain how a discrete valuation υ on a field $\mathbb{F}$ gives rise to a non-trivial action of the matrix group $SL(2,\mathbb{F})$ on a simplicial tree $T(\upsilon)$. Consequently, any representation of a 3-manifold group with values in $SL(2,\mathbb{F})$ leads to an action of that group on $T(\upsilon)$, and once it is understood when the action is non-trivial, to essential surfaces.

Two methods of constructing such actions are described here. The first, found in §3, uses the fact that if M is an orientable, finite volume, complete hyperbolic 3-manifold, there is a holonomy representation $\pi_1(M) \to SL(2,\mathbb{F})$, where $\mathbb{F} \subset \mathbb{C}$ is a number field. As a finite extension of the rationals, $\mathbb{F}$ admits a rich family of discrete valuations and consequently, actions of $\pi_1(M)$ on simplicial trees.

The second method, due to Culler and Shalen, is developed over the next three sections of this chapter. Section 4 describes how to endow the set $R(\pi)$ of $SL(2,\mathbb{C})$-representations of a finitely generated group π with the structure of a complex affine algebraic set, while §5 does the same for the set $X(\pi)$ of characters of such representations. The algebraic geometry of curves is invoked in §6 to identify each point x of the smooth projective model of an algebraic curve $X_0 \subset X(\pi)$ with a discrete valuation on its function field $\mathbb{C}(X_0)$. This leads to an action of π on a simplicial tree which turns out to be non-trivial precisely when x is an ideal point.

The seventh section of the chapter describes Culler and Shalen's remarkable quantification of the work of the previous three sections. More precisely, a detailed analysis of the actions of the fundamental group of a knot manifold M on the simplicial trees determined by ideal points of a curve X_0 in $X(\pi_1(M))$ provides a refined understanding of the associated essential surfaces. This analysis leads to the definition of the *Culler-Shalen seminorm* on $H_1(\partial M;\mathbb{R})$ determined by X_0. Further, when M is hyperbolic and the curve is chosen to contain the character of a holonomy representation of M, this seminorm is actually a norm whose unit ball is a finite-sided polygon which carries significant topological information about M.

The proof of the cyclic surgery theorem is the focus of §8. In the case that the knot manifold M is hyperbolic and the curve X_0 in $X(\pi_1(M))$ is chosen to contain the character of a holonomy representation, we invoke the hypotheses of the theorem in two seemingly banal ways. First, manifolds with cyclic fundamental groups do not contain closed essential surfaces of positive genus, and second, the image of any $SL(2,\mathbb{C})$-representation of a cyclic group is cyclic. It turns out that the first of these observations provides significant information about the ideal points of X_0, while the second provides information about its non-ideal points. Taken together, strong restrictions are obtained on the values of the Culler-Shalen norm of X_0 on the classes in $H_1(\partial M;\mathbb{R})$ corresponding to the cyclic surgery slopes, and these restrictions lead to the proof of the cyclic surgery theorem in the case under consideration.

The final sections discuss further applications of character variety methods to surgery theory and the technical developments needed to achieve them. Section 9 outlines the proof of the finite surgery theorem, which states that there are at most five slopes α on the boundary of a hyperbolic knot manifold for which $\pi_1(M(\alpha))$ is finite and the distance between any two such slopes is at most 3. All of the elements which arise in the proof of the cyclic surgery theorem occur here as well, though the character variety methods need to be supplemented with a refinement of Culler-Shalen norms coming from the A-polynomial, a character variety invariant introduced by Cooper, Culler, Gillet, Long and Shalen in 1994 ([CCGLS]). We discuss this invariant and describe how it is used in the proof of the finite surgery theorem in this section.

Section 10 shows how to obtain bounds on the distances between finite surgery slopes and slopes whose associated Dehn fillings have positive dimensional character varieties. The latter occurs in topologically interesting situations, and we discuss applications to surgeries yielding reducible or Seifert fibred manifolds.

The final section, §11, begins with a brief discussion of the relationship between distance bounds and the geography of the set $\mathcal{E}(M)$ of exceptional filling slopes of a hyperbolic knot manifold M, i.e., those whose associated fillings are not hyperbolic manifolds. We then give a selective survey of the known distance bounds between the subsets of $\mathcal{E}(M)$ corresponding to reducible manifolds, manifolds with cyclic or finite fundamental groups, small Seifert manifolds, and toroidal manifolds.

We chart a direct path to our goals and refer the reader to Peter Shalen's handbook article [Sh2] for a broader, more detailed discussion of the use of representation-theoretic methods in 3-manifold topology, up to the year 2000. His paper [Sh1], which discusses the historical evolution of many of the ideas presented here, is highly recommended.

Many thanks to the organisers of the ICERM Topical workshop **Perspectives on Dehn Surgery** for putting together this exciting event. Special thanks go to Michelle Chu, who served as TA for the minicourse and whose numerous suggestions greatly improved the lectures. The author also thanks the referees and the editors of this volume for their input, as well as Giacomo Bascapè, Duncan McCoy, Patricia Sorya, and Sarah Zbida for their comments on a preliminary version of the notes.

2. From surfaces to actions, and back again

Throughout these notes we assume that 3-manifolds are connected, orientable, and irreducible.

A *knot manifold* is a compact 3-manifold (and, by hypothesis, connected, orientable, irreducible) with incompressible torus boundary. For instance, if K is a non-trivial knot in the 3-sphere with closed tubular neighbourhood $N(K)$, then the knot's *exterior*

$$M_K = S^3 \setminus \mathrm{int}(N(K))$$

is a knot manifold. More generally if K is a knot in a closed 3-manifold W which does not bound a disk in W, its *exterior* $M_K = W \setminus \mathrm{int}(N(K))$ is a knot manifold provided that K is not contained in a 3-ball in W (to guarantee irreducibility).

Surfaces S in 3-manifolds W will be assumed to be orientable and *properly embedded*: S is a closed subset of W, transverse to ∂W, and satisfies $S \cap \partial W = \partial S$. We also assume that surfaces have *bicollar neighbourhoods*, meaning that there is a neighbourhood $N(S)$ of S in W where

$$(N(S), N(S) \cap \partial W, S) \cong (S \times [-1,1], \partial S \times [-1,1], S \times \{0\})$$

A *graph* is a 1-dimensional CW complex and a *tree* is a contractible graph. The *vertex set* of a tree T is its 0-skeleton V. Its *edge set* is the family of closed subintervals obtained as the closures of the components of $T \setminus V$.

We consider trees to be the underlying spaces of simplicial complexes and use $\mathrm{Aut}(T)$ to denote the group of simplicial homeomorphisms of T. The set of midpoints of the edges of T will be denoted E.

In the first section of this chapter we describe how a surface in a 3-manifold M gives rise to an action of $\pi_1(M)$ on a simplicial tree. Following Stallings [St], we reverse the construction in the second section by showing how such actions leads to essential surfaces in M.

2.1. **Surfaces to actions.** Suppose that S is a surface in a 3-manifold M with bicollar $N(S)$. Our goal is to construct an associated tree T and an action of $\pi_1(M)$ on T which determines S.

Let $p : \widetilde{M} \to M$ be a universal cover and consider the inverse image $\widetilde{S} = p^{-1}(S)$ of S in $\widetilde{M}$. It is straightforward to verify that $\widetilde{S}$ is orientable, properly embedded, and has a bicollar neighbourhood $N(\widetilde{S}) = p^{-1}(N(S))$ in $\widetilde{M}$. The following exercise shows that the components of $\widetilde{S}$ are separating (i.e., their complements are not connected).

Exercise 2.1. Suppose that F is a non-separating surface in a 3-manifold W. Show that there is an infinite cyclic covering space $\widehat{W} \to W$ built by alternately piecing together copies of a bicollar neighbourhood $N(F)$ of F in W and copies of the connected set $W \setminus \mathrm{int}(N(F))$. Deduce that W is not simply-connected.

We can construct a graph from $\widetilde{S}$ as follows:

- the vertex set V is the set of components of $\widetilde{M} \setminus N(\widetilde{S})$;
- the edge set is the set of components of $\widetilde{S}$, where the vertices of an edge are the components of $\widetilde{M} \setminus N(\widetilde{S})$ lying to either side of it.

It is straightforward to see that this graph is a tree T using the exercise above.

The deck transformation action of the fundamental group of M on $\widetilde{M}$ preserves the vertex and edge sets of T and in this way determines an action of $\pi_1(M)$ on T by simplicial automorphisms. A useful feature of this action is that it is *without inversions*. In other words, it satisfies the conclusion of the following exercise.

Exercise 2.2. Show that if $\gamma \in \pi_1(M)$ fixes an edge of T, then it fixes the vertices of T incident to the edge.

We can realise the action topologically through the construction of a continuous map $f : \widetilde{M} \to T$ which converts the deck transformation action of $\pi_1(M)$ on $\widetilde{M}$ into its action on T. More precisely, define $f|_{\widetilde{M} \setminus N(\widetilde{S})}$ so that it sends each component of $\widetilde{M} \setminus N(\widetilde{S})$ to the corresponding vertex of T. Then $f(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot f(\tilde{x})$ is easily seen to hold for $\gamma \in \pi_1(M)$ and $\tilde{x} \in \widetilde{M} \setminus N(\widetilde{S})$.

Now extend $f|_{\widetilde{M} \setminus N(\widetilde{S})}$ over each component $N(\widetilde{S}_0) \cong \widetilde{S}_0 \times [-1,1]$ of $N(\widetilde{S})$ by composing the projection to $[-1,1]$ with a simplicial homeomorphism between $[-1,1]$ and the edge corresponding to that component. The identity $f(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot f(\tilde{x})$ will continue to hold as long as we choose the homeomorphism $N(\widetilde{S}) \cong \widetilde{S} \times [-1,1]$ to be an appropriate lift of a homeomorphism $N(S) \cong S \times [-1,1]$.

By construction, $\widetilde{S} = f^{-1}(E)$, where E is the set of midpoints of the edges of T. Therefore $S = f^{-1}(E)/\pi_1(M)$. Summarising,

$$S \subset M \quad \longmapsto \quad \widetilde{S} \subset \widetilde{M} \quad \longmapsto \quad \begin{cases} \text{a simplicial tree } T \text{ on which} \\ \pi_1(M) \text{ acts simplicially without} \\ \text{inversions and an equivariant} \\ \text{map } \widetilde{M} \to T \text{ which determines } S \end{cases}$$

2.2. **Actions to surfaces.** Here we reverse the construction of the previous section.

Proposition 2.3. *Suppose that $\pi_1(M)$ acts simplicially and without inversions on a tree T. Then there is an equivariant map $f : \widetilde{M} \to T$ such that $S = f^{-1}(E)/\pi_1(M)$ is a properly embedded surface in M.*

Proof. Fix a simplicial triangulation on M, lift it to $\widetilde{M}$, and denote the i-skeleton of the lifted triangulation by $\widetilde{M}^{(i)}$. The simple-connectivity of the simplices of M implies that $\pi_1(M)$ acts freely on the set of i-simplices of $\widetilde{M}$. We will use this observation to successively construct equivariant maps $f^{(i)} : \widetilde{M}^{(i)} \to T$ for $i = 0, 1, 2, 3$ and take $f = f^{(3)}$. We use $\{v_j\}$, $\{e_k\}$, and $\{\tau_l\}$ to denote complete systems of $\pi_1(M)$-orbit representatives of the vertices, edges, and 2-simplices of $\widetilde{M}$.

We begin with $f^{(0)}$. For each j, take $f^{(0)}(v_j)$ to be an arbitrarily chosen vertex of T. Since $\pi_1(M)$ acts freely on $\widetilde{M}^{(0)}$, each vertex v of $\widetilde{M}$ can be expressed uniquely in the form $\gamma \cdot v_j$ for some j and $\gamma \in \pi_1(M)$, so setting $f^{(0)}(v) = \rho(\gamma)(f^{(0)}(v_j))$ defines an equivariant map $f^{(0)} : \widetilde{M}^{(0)} \to T$.

To define $f^{(1)}$, fix one of the edges e_k. If $f^{(0)}(\partial e_k)$ is a single point, we take $f^{(1)}|_{e_k}$ to be constantly that point. Otherwise we define $f^{(1)}|_{e_k}$ so that its image is the simplicial geodesic connecting the vertices of $f^{(0)}(\partial e_k)$. As in the previous paragraph, setting $f^{(1)}|_{\gamma \cdot e_k} = \rho(\gamma) \circ (f^{(1)}|_{e_k})$ and repeating the construction for each k determines an equivariant map $f^{(1)} : \widetilde{M}^{(1)} \to T$ which extends $f^{(0)}$.

The definition of $f^{(2)}$ is analogous: For each τ_l we extend $f^{(1)}|_{\partial \tau_l}$ across τ_l using the contractibility of T and define $f^{(2)}|_{\gamma \cdot \tau_l} = \rho(\gamma) \circ (f^{(2)}|_{\tau_l})$. A similar approach yields $f^{(3)} = f$.

To produce the surface S, we refine the construction by requiring that each $f^{(i)}$ be transverse to the set E of midpoints of the edges of T. We will go a bit further by arranging for f to be a simplicial map, at least after possibly subdividing the triangulation of M.

To see how this is done, first note that $f^{(0)}$ is already simplicial. Next subdivide each edge representative e_k and choose the extension of $f^{(0)}|_{\partial e_k}$ over e_k so that $f^{(1)}|_{e_k}$ is simplicial. After extending these subdivisions and each $f^{(1)}|_{e_k}$ equivariantly across $\widetilde{M}^{(1)}$, $f^{(1)}$ becomes a simplicial map.

Next consider $\widetilde{M}^{(2)}$. For each 2-simplex τ_l we can apply the simplicial approximation theorem to produce a subdivision of τ_l extending that on $\partial\tau_l$ and an extension of $f^{(1)}|_{\partial\tau_l}$ over τ_l which is simplicial with respect to the subdivision ([Spa, Lemma 1 and Theorem 8, Section 3, Chapter 4]). Doing this for each l and extending equivariantly yields the desired subdivison of $\widetilde{M}^{(2)}$ and simplicial $f^{(2)}$. Similarly we obtain an equivariant subdivision of $\widetilde{M} = \widetilde{M}^{(3)}$ and an $f = f^{(3)}$ which is simplicial. Since the subdivision of $\widetilde{M}$ is invariant under the group of deck transformations of the cover $\widetilde{M} \to M$, it is the pull-back of a subdivision of the triangulation on M.

Assume then that f is simplicial and let σ be a 3-simplex of $\widetilde{M}$. If x is the mid-point of an edge e of T, then $f(\sigma) \cap e$ is either empty or all of e. In the first case $(f|_\sigma)^{-1}(x) = \emptyset$ and in the second it is a linear 2-disk slice of σ. It follows that $\widetilde{S} = f^{-1}(E)$ is a 2-manifold properly embedded in $\widetilde{M}$ whose intersection with each 3-simplex σ of $\widetilde{M}$ is either empty or a properly embedded disk disjoint from the vertices of σ. The equivariance of f shows that $\widetilde{S}$ is invariant under the group of deck transformations and therefore projects to a compact surface S properly embedded in M.

Since $\pi_1(M)$ acts without inversions on T, we can define a $\pi_1(M)$-invariant orientation of the edges of T by arbitrarily orienting a complete set of edge representatives of the $\pi_1(M)$ action on T and transporting these orientations across T using the $\pi_1(M)$ action. Pulling back these orientations by f determines a $\pi_1(M)$-invariant transverse orientation of $\widetilde{S}$, which descends to one on S. Hence S is orientable. As an orientable PL surface in M, it is also bicollared. $\square$

Remark 2.4. The construction of the surface S works equally well when f is only required to be transverse to E. This added flexibility will be useful below.

We say that a surface $S \subset M$ constructed from an equivariant map $f : \widetilde{M} \to T$ which is transverse to E is *dual* to the action of $\pi_1(M)$ on the tree T. We end this section by showing that under a mild added hypothesis, there is a dual surface which is non-empty and essential.

Non-emptiness is easy to arrange, for if the empty set is dual to an action, the connected set $f(\widetilde{M})$ is disjoint from E. Then $f(\widetilde{M})$ contains a unique vertex of T which, by construction, must be fixed by every element of $\pi_1(M)$.

Definition 2.5. A simplicial action without inversions of a group G on a tree T is *non-trivial* if no vertex of T is fixed by G.

No dual surface to a non-trivial action without inversions can be empty.

Recall that a surface S in a 3-manifold M is *essential* if it is orientable, properly embedded, incompressible, boundary incompressible, and has no components which are either boundary-parallel or 2-spheres bounding a 3-balls in M.

Theorem 2.6. (Stallings) *If $\pi_1(M)$ acts non-trivially and without inversions on a tree T, there is a non-empty essential surface S in M dual to the action.*

Proof. Our strategy is to begin with an arbitrary dual surface S and to modify the associated equivariant map $f : \widetilde{M} \to T$ to yield a new one $f' : \widetilde{M} \to T$ whose dual surface is essential. We'll only sketch the proof and refer the reader to [Sh2] for more complete details. Better still, the reader can work out the details for themselves.

Start with a dual surface $S = f^{-1}(E)/\pi_1(M)$ for an equivariant map $f : \widetilde{M} \to T$ which is transverse to E. If S is not essential, then one or more of the following situations arises:

(1) S has a boundary-parallel component S_0;

(2) S has a 2-sphere component S_0;

(3) some component S_0 of S admits a compressing disk.

(4) some component S_0 of S admits a boundary-compressing disk.

Since incompressible non-boundary parallel surfaces in knot manifolds admit no boundary-compressing disks ([Ha2, Lemma 1.10]), it suffices to alter f so that the first three cases do not arise.

In the first case, S_0 cobounds a product region P with a subsurface of ∂M. After shrinking $M \setminus (P \setminus S_0)$ in a collar of S_0, we obtain a submanifold M_0 of M which is a strong deformation retract of M such that $M_0 \cap S$ is a union of components of $S \setminus S_0$. There is a homeomorphism $r : M \to M_0$, supported in an arbitrarily small neighbourhood of $M \setminus \mathrm{int}(M_0)$, such that the composition $M \xrightarrow{r} M_0 \subset M$ lifts to a $\pi_1(M)$-equivariant map $\tilde{r} : \widetilde{M} \to \widetilde{M}$. Then $f \circ \tilde{r} : \widetilde{M} \to T$ is an equivariant map transverse to E whose associated dual surface is $M_0 \cap S$. After finitely many such modifications, we can suppose that S has no boundary-parallel components.

Suppose next that S has a 2-sphere component S_0. Since M is irreducible, $S_0 = \partial B_0$, where B_0 is a 3-ball in M. We can thicken B_0 a little to obtain a 3-ball $B \supset B_0$ such that $B \cap S = B_0 \cap S$. The lift of B to $\widetilde{M}$ is a disjoint union of 3-balls on which $\pi_1(M)$ acts freely and transitively. Fix one such lift $\widetilde{B}$, and let $\widetilde{S}_0$ be the component of $p^{-1}(S_0)$ it contains. If e is the edge of T whose midpoint y_0 is $f(\widetilde{S}_0)$, we can assume that $f(\partial \widetilde{B}) = \{y\} \subset \mathrm{int}(e) \setminus \{y_0\}$. Define $f' : \widetilde{M} \to T$ so that

$$f'(x) = \begin{cases} f(x) & \text{if } x \in \widetilde{M} \setminus p^{-1}(B) \\ \gamma \cdot y & \text{if } x \in \gamma \cdot \widetilde{B} \end{cases}$$

Then f' is equivariant and transverse to E. Further, $f_1^{-1}(E) \subseteq \widetilde{S} \setminus p^{-1}(S_0)$, so that if S' is the associated dual surface, then $S' \subseteq S \setminus S_0$. Again, after finitely many such operations we can suppose that S has no 2-sphere components.

Finally, suppose that some component S_0 of S admits a compressing disk D. Without loss of generality we can assume that $D \cap S = \partial D \subset S_0$. Let B be a 3-ball neighbourhood of D in M such that $B \cap S$ is an annular neighbourhood of ∂D in S_0 which divides B into a solid torus to one side of S_0 and a 3-ball to the other. One can alter f on the lifts of B to $\widetilde{M}$ so that the dual surface S' is obtained by surgering S along D. Remove any boundary-parallel or 2-sphere components of S', as above, and if the resulting surface admits a compressing disk, we can surger it away. Eventually, the process terminates to yield an essential surface, dual to the action, which is necessarily non-empty by non-triviality. This completes the proof. □

Remark 2.7. Bass-Serre theory codifies the structure of groups which act non-trivially and simplicially on a tree ([Se]). It shows, for instance, that a group acts non-trivially and simplicially on a tree if and only if it admits a non-trivial splitting as either a free product with amalgamation or an HNN extension. Combined with the work of Stallings discussed above, it implies that the fundamental group of a 3-manifold which contains no essential surfaces admits no non-trivial actions on simplicial trees.

3. The SL_2-tree of a discrete valuation

The first two parts of this section detail a construction of Bruhat and Tits which associates a simplicial tree $T(\upsilon)$ to a discrete valuation υ on a field $\mathbb{F}$ and a simplicial action of $SL(2, \mathbb{F})$ on $T(\upsilon)$. A standard reference is [Se]. The last part describes one way in which this construction can be used to construct essential surfaces in 3-manifolds.

3.1. Discrete valuation rings.

3.1. Discrete valuation rings. A (discrete) *valuation* on a field $\mathbb{F}$ is a surjective homomorphism $\upsilon : \mathbb{F}^* = \mathbb{F} \setminus \{0\} \to \mathbb{Z}$ satisfying

$$\upsilon(a+b) \geq \min\{\upsilon(a), \upsilon(b)\}$$

whenever $a, b, a+b \in \mathbb{F}^*$. A key associated object is the *valuation ring*

$$\mathcal{O}(\upsilon) = \{0\} \cup \{a \in \mathbb{F}^* \mid \upsilon(a) \geq 0\},$$

which is a subring of $\mathbb{F}$ whose group of units is readily determined: A non-zero element a of $\mathbb{F}$ is a unit of $\mathcal{O}(\upsilon)$ if and only if $\upsilon(a) \geq 0$ and $-\upsilon(a) = \upsilon(a^{-1}) \geq 0$. That is, if and only if $\upsilon(a) = 0$. It follows that the ideal $\mathfrak{M}(\upsilon) = \{0\} \cup \{a \in \mathbb{F} \mid \upsilon(a) \geq 1\}$ of $\mathcal{O}(\upsilon)$ is maximal. If π denotes an element of $\mathcal{O}(\upsilon)$ with $\upsilon(\pi) = 1$, then each element $x \in \mathcal{O}(\upsilon)$ can be written as $\pi^n u$, where $n = \upsilon(x)$ and u is a unit of $\mathcal{O}(\upsilon)$. It is easy to verify that $\mathfrak{M}(\upsilon) = (\pi)$.

Exercise 3.1. Show that $\mathcal{I}$ is an ideal in $\mathcal{O}(\upsilon)$ if and only if $\mathcal{I} = (\pi^n)$ for some integer $n \geq 0$.

Hence $\mathcal{O}(\upsilon)$ is a PID whose ideals form a nested sequence

$$\{0\} \subset \cdots \subset (\pi^{n+1}) \subset (\pi^n) \subset \cdots \subset (\pi^2) \subset (\pi) = \mathfrak{M}(\upsilon) \subset (1) = \mathcal{O}(\upsilon)$$

Example 3.2. Let $U \subseteq \mathbb{C}$ be an open set. A function $f : U \to \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$ is called *meromorphic* if it is holomorphic with respect to the complex structure on $\mathbb{C}P^1$. The set of meromorphic functions on $\mathbb{C}$ forms a field $\mathbb{F}$ where addition and multiplication are defined pointwise. Each $z \in \mathbb{C}$ determines a discrete valuation $\upsilon_z : \mathbb{F} \setminus \{0\} \to \mathbb{Z}$, where

$$\upsilon_z(f) = n \quad \text{if} \quad \begin{cases} f(z) \in \mathbb{C} \text{ and } f \text{ has a zero of order } n \geq 0 \text{ at } z \\ \\ f(z) = \infty \text{ and } f \text{ has a pole of order } -n > 0 \text{ at } z \end{cases}$$

The valuation ring $\mathcal{O}(\upsilon_z)$ is the ring of meromorphic functions on U which are holomorphic (i.e., complex valued) at z and $\mathfrak{M}(\upsilon_z)$ is the set of meromorphic functions on U which have a zero at z.

3.2. The tree of a valuation. To each valuation υ on a field $\mathbb{F}$ we can associate a simplicial tree $T(\upsilon)$ as follows.

The vertices of $T(\upsilon)$ are classes of lattices in $\mathbb{F}^2$, where by lattice we mean an $\mathcal{O}(\upsilon)$-submodule $L \subset \mathbb{F}^2$ which generates $\mathbb{F}^2$ as an $\mathbb{F}$-vector space. Lattices L, L' are considered equivalent if $L' = xL$ for some $x \in \mathbb{F}^*$, and we denote the set of such classes by V. Since each $x \in \mathbb{F}^*$ can be written $\pi^n u$, where $n \in \mathbb{Z}$ and u is a unit of $\mathcal{O}(\upsilon)$, the class of L is given by $\Lambda = \{\pi^n L \mid n \in \mathbb{Z}\}$.

Before describing the edge set of $T(\upsilon)$, we make some observations.

Since $\mathcal{O}(\upsilon)$ is a PID, each lattice is isomorphic to $\mathcal{O}(\upsilon)^2$, and from this it is easy to see that if L, L' are lattices, there is some $A \in GL(2, \mathbb{F})$ for which $L' = A(L)$. For $k \gg 0$, the coefficients of the matrix $\pi^k A$ lie in $\mathcal{O}(\upsilon)$ and therefore $\pi^k L' = (\pi^k A)(L) \subseteq L$. Again using the fact that $\mathcal{O}(\upsilon)$ is a PID, we can apply Smith normal form to see that there are integers $0 \leq a \leq b$ and a basis $\{e_1, e_2\}$ of L such that $\{\pi^a e_1, \pi^b e_2\}$ is a basis of $\pi^k L'$. Then $m = k - a$ is the smallest integer j for which $\pi^j L' \subseteq L$:

$$\pi^m L' = (\pi^m A)(L) \subseteq L \quad \text{and} \quad \pi^m L' \not\subseteq \pi L$$

Put another way, $\pi^m L'$ is the largest representative of the class of L' contained in L. By construction,

$$L/\pi^m L' \cong \mathcal{O}(\upsilon)/(\pi^n)$$

where $n = b - a \geq 0$. We leave it as an exercise to show that replacing L and L' by any other representatives of their classes Λ, Λ' and repeating the process, we end up with the same quotient $\mathcal{O}(\upsilon)/(\pi^n)$. Thus we can unambiguously define

$$d(\Lambda, \Lambda') = n$$

It is evident that $d(\Lambda, \Lambda') = 0$ if and only if $\Lambda = \Lambda'$. We connect classes Λ, Λ' with an edge if and only if $d(\Lambda, \Lambda') = 1$ and let $T(\upsilon)$ be the resultant graph.

Proposition 3.3. *$T(\upsilon)$ is a tree.*

Proof. We must show that $T(\upsilon)$ is simply-connected. Equivalently, $T(\upsilon)$ is path-connected and contains no non-trivial cycles.

Let $\Lambda, \Lambda' \in V$ be distinct and choose representative lattices $L \supseteq L'$ for them so that $L/L' \cong \mathcal{O}(\upsilon)/(\pi^n)$ for some $n \geq 1$. For $0 \leq i \leq n$, denote by L_i the inverse image in L of the submodule $(\pi^i L)/L')$ of L/L'. Then L_i is a lattice with $L_0 = L$ and $L_n = L'$. Moreover for $0 \leq i \leq n-1$, the quotient L_i/L_{i+1} is isomorphic to $\mathcal{O}(\upsilon)/(\pi)$, so if Λ_i is the class of L_i, then $d(\Lambda_i, \Lambda_{i+1}) = 1$. It follows that $T(\upsilon)$ is path-connected.

Next we show that $T(\upsilon)$ contains no non-trivial cycles. That is, if $n \geq 1$ and $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ is a sequence of vertices with $d(\Lambda_i, \Lambda_{i+1}) = 1$ and no backtracks, then $\Lambda_0 \neq \Lambda_n$.

Choose representative lattices L_i for Λ_i for which $L_n \subset L_{n-1} \subset \cdots \subset L_1 \subset L_0$ and $L_i/L_{i+1} \cong \mathcal{O}(\upsilon)/(\pi)$ for each i. We assume inductively that $L_0/L_{n-1} \cong \mathcal{O}(\upsilon)/(\pi^{n-1})$, where $n \geq 2$, and will show that $L_0/L_n \cong \mathcal{O}(\upsilon)/(\pi^n)$. This implies that $d(\Lambda_0, \Lambda_n) = n$, so $\Lambda_0 \neq \Lambda_n$.

By assumption, $\pi L_{n-2} \subset L_{n-1}$ with $L_{n-1}/\pi L_{n-2} \cong \mathcal{O}(\upsilon)/(\pi) \cong L_{n-1}/L_n$. Hence if $\pi L_{n-2} \subseteq L_n$ then $\pi L_{n-2} = L_n$, contrary to our no backtrack hypothesis. Thus $L_n \subsetneq L_n + \pi L_{n-2} \subseteq L_{n-1}$, and as L_{n-1}/L_n is a simple $\mathcal{O}(\upsilon)$ module we have $L_{n-1} = L_n + \pi L_{n-2}$. Our assumption that $L_0/L_{n-1} \cong \mathcal{O}(\upsilon)/(\pi^{n-1})$ implies that $L_n + \pi L_{n-2} = L_{n-1} \not\subseteq \pi L_0$. As such, $L_n \not\subseteq \pi L_0$ so that L_n is the maximal representative of Λ_n contained in L_0. Thus $L_0/L_n \cong \mathcal{O}/(\pi^d)$, where $d = d(\Lambda_0, \Lambda_n)$. Consideration of the exact sequence $0 \to L_{n-1}/L_n \to L_0/L_n \to L_0/L_{n-1} \to 0$ then shows that $n = d$. $\square$

3.3. **The SL_2-action.** The natural action of $GL(2, \mathbb{F})$ on $\mathbb{F}^2$ preserves the set of lattices and descends to a transitive action on the vertex set V of $T(\upsilon)$. Further, if $A \in GL(2, \mathbb{F})$ and $L' \subset L$ are lattices for which $L/L' \cong \mathcal{O}(\upsilon)/(\pi)$, then $A(L') \subset A(L)$ and $A(L)/A(L') \cong \mathcal{O}(\upsilon)/(\pi)$. Thus the action of $GL(2, \mathbb{F})$ on V extends to one on $T(\upsilon)$. We focus on the restriction of this action to $SL(2, \mathbb{F})$.

Proposition 3.4. *Let υ be a discrete valuation on a field $\mathbb{F}$ and $T(\upsilon)$ the associated tree.*

(1) *The $SL(2, \mathbb{F})$-stabiliser of a vertex of $T(\upsilon)$ is conjugate over $GL(2, \mathbb{F})$ to $SL(2, \mathcal{O}(\upsilon))$. Further, an element $A \in SL(2, \mathbb{F})$ fixes a vertex of $T(\upsilon)$ if and only if its trace lies in $\mathcal{O}(\upsilon)$.*

(2) $SL(2,\mathbb{F})$ *acts on* $T(v)$ *without inversions. That is, if* $A \in SL(2,\mathbb{F})$ *leaves an edge* e *of* $T(v)$ *invariant, it fixes the vertices incident to* e.

Proof. (1) If $A \in SL(2,\mathbb{F})$ and $B \in GL(2,\mathbb{F})$, then A fixes a class $\Lambda \in V$ if and only if BAB^{-1} fixes $B \cdot \Lambda$. Hence the fact that $GL(2,\mathbb{F})$ acts transitively on V implies that any vertex stabiliser is conjugate over $GL(2,\mathbb{F})$ to the stabiliser of the class Λ_0 of the lattice $\mathcal{O}(v)^2 \subset \mathbb{F}^2$. Moreover, any conjugate of this stabiliser is the stabiliser of some vertex. We'll show that the stabiliser of Λ_0 is $SL(2,\mathcal{O}(v))$.

It is clear that $SL(2,\mathcal{O}(v))$ fixes Λ_0. Conversely, if $A \in SL(2,\mathbb{F})$ fixes Λ_0, then $A \cdot \mathcal{O}(v)^2 = \pi^m \mathcal{O}(v)^2$ for some integer m; therefore $\pi^{-m}A$ is an automorphism of $\mathcal{O}(v)^2$. Then $\det(\pi^{-m}A)$ is a unit of $\mathcal{O}(v)$ so that $0 = v(\det(\pi^{-m}A)) = -2m$. Hence A is an automorphism of $\mathcal{O}(v)^2$, so lies in $SL(2,\mathcal{O}(v))$, which proves the first assertion of (1).

For the second assertion, note that we have just seen that $\mathrm{tr(A)} \in \mathcal{O}(v)$ if A fixes a vertex of $T(v)$. Conversely suppose that $\mathrm{tr(A)} \in \mathcal{O}(v)$. We show that A is conjugate in $GL(2,\mathbb{F})$ to an element of $SL(2,\mathcal{O}(v))$, and so as any $GL(2,\mathbb{F})$-conjugate of $SL(2,\mathcal{O}(v))$ is the stabiliser of some vertex, A fixes a vertex.

In the case that there is an element $v \in \mathbb{F}^2$ such that the pair $\{v, A(v)\}$ spans $\mathbb{F}^2$, the Cayley-Hamilton theorem implies that A is conjugate to a matrix of the form $\begin{pmatrix} 0 & -1 \\ 1 & \mathrm{tr}(A) \end{pmatrix} \in SL(2,\mathcal{O}(v))$. Thus A fixes a vertex of $T(v)$. On the other hand, if $A(v)$ is a multiple of v for each v, then the linearity of A implies that it acts on $\mathbb{F}^2$ as multiplication by some $x \in \mathbb{F}^*$. Since $1 = \det(A) = x^2$, we have $x = \pm 1$ and therefore $A = \pm I \in SL(2,\mathcal{O}(v))$, which acts as the identity on $T(v)$.

(2) Let $A \in SL(2,\mathbb{F}), \Lambda \in V$ and L a lattice representing Λ. It follows from our discussions in §3.2 that there are integers $a \leq b$ and a basis $\{e_1, e_2\}$ of L such that $\{\pi^a e_1, \pi^b e_2\}$ is a basis of $L' = A(L)$. From this it is easy see that

$$d(\Lambda, A(\Lambda)) = b - a$$

We also have $L' = A'(L)$ where $A' = \begin{pmatrix} \pi^a & 0 \\ 0 & \pi^b \end{pmatrix}$, so $A^{-1}A'$ is an automorphism of L whose determinant is a unit of $\mathcal{O}(v)$. Then $0 = v(\det(A^{-1}A')) = v(\det(A')) - v(\det(A)) = v(\det(A'))$. Consequently,

$$d(\Lambda, A(\Lambda)) \equiv a + b = v(\det(A')) = 0 \pmod 2$$

In other words, $d(\Lambda, A(\Lambda))$ is even for each $A \in SL(2,\mathbb{F})$ and $\Lambda \in V$. It follows that if A leaves an edge of $T(v)$ invariant, it fixes the vertices of the edge. This completes the proof. □

It is an immediate consequence of the proposition that $SL(2,\mathbb{F})$ acts non-trivially on $T(v)$. Indeed, since the action is without inversions we deduce:

Corollary 3.5. *An element of* A *of* $SL(2,\mathbb{F})$ *acts without fixed points on* $T(v)$ *if and only if* $v(tr(A)) < 0$. □

Remark 3.6. Since $-I \in SL(2,\mathbb{F})$ acts as the identity on $T(v)$, the $SL(2,\mathbb{F})$-action on $T(v)$ factors through a $PSL(2,\mathbb{F})$-action and in this context, analogues of Proposition 3.4 and Corollary 3.5 hold with $SL(2,\mathcal{O}(v))$ and $SL(2,\mathbb{F})$ replaced by $PSL(2,\mathcal{O}(v))$ and $PSL(2,\mathbb{F})$. Of course, the trace of an element of $PSL(2,\mathbb{F})$ is only defined up to sign, but this ambiguity has no impact on the statements of the results or their proofs.

3.4. **Valuations and essential surfaces.** Here is an immediate consequence of Theorem 2.6 and Corollary 3.5.

Theorem 3.7. *Let M be a 3-manifold, υ a discrete valuation on a field $\mathbb{F}$, and $\rho : \pi_1(M^3) \to SL(2, \mathbb{F})$ a homomorphism whose image contains an element A for which $\upsilon(tr(A)) < 0$. Then $\pi_1(M)$ acts non-trivially on $T(\upsilon)$. In particular, M contains an essential surface.* □

Corollary 3.8. *Suppose that M is a 3-manifold with finitely generated fundamental group and which contains no essential surfaces. If υ a discrete valuation on a field $\mathbb{F}$ and $\rho : \pi_1(M) \to SL(2, \mathbb{F})$ a homomorphism, then ρ is conjugate over $GL(2, \mathbb{F})$ to a representation with values in $SL(2, \mathcal{O}(\upsilon))$.*

Proof. Theorem 3.7 implies that $\upsilon(\mathrm{tr}(\rho(\gamma))) \geq 0$ for each $\gamma \in \pi_1(M)$, so every element of $\rho(\pi_1(M))$ fixes a vertex of $T(\upsilon)$ by Proposition 3.4. Applying part (6) of Exercise 3.11 (below) implies that $\pi_1(M)$ fixes a vertex f $T(\upsilon)$ and therefore Proposition 3.4 shows that ρ is conjugate over over $GL(2, \mathbb{F})$ to a representation with values in $SL(2, \mathcal{O}(\upsilon))$. □

Remark 3.9. As in Remark 3.6, the obvious analogues of Theorem 3.7 and Corollary 3.8 hold with $SL(2, \mathbb{F})$ and $SL(2, \mathcal{O}(\upsilon))$ replaced by $PSL(2, \mathbb{F})$ and $PSL(2, \mathcal{O}(\upsilon))$.

Examples 3.10.

(1) Suppose that M is a complete finite volume oriented hyperbolic 3-manifold. The hyperbolic structure on M determines a *holonomy representation*

$$\rho_M : \pi_1(M) \to PSL(2, \mathbb{C}) \cong \mathrm{Isom}_+(\mathbb{H}^3)$$

which is faithful, has discrete image, and is unique up to conjugation in $\mathrm{Isom}(\mathbb{H}^3)$ by the Mostow-Prasad rigidity theorem ([MR, Theorem 1.6.3]). This representation determines M in that

$$\mathrm{int}(M) \cong \mathbb{H}^3/\rho_M(\pi_1(M))$$

It is a further consequence of Mostow-Prasad rigidity that ρ_M is conjugate to a representation with values in $PSL(2, \mathbb{F})$, where $\mathbb{F}$ is a finite extension field of $\mathbb{Q}$ (see [MR, Theorem 3.1.2]).

Suppose that there is a $\gamma \in \pi_1(M)$ such that $\pm\mathrm{tr}(\rho_M(\gamma))$ is not an algebraic integer. A basic result in commutative algebra states that the ring of integers of $\mathbb{F}$ is the intersection of all the discrete valuation rings in $\mathbb{F}$ (see [Ba, Lemma 6.8(2)]), so there is a discrete valuation υ on $\mathbb{F}$ such that $\upsilon(\mathrm{tr}(\rho_M(\gamma))) < 0$. Taking Remark 3.9 into account, Theorem 3.7 implies that M contains an essential surface.

The same argument associates essential surfaces in M to any representation $\rho : \pi_1(M) \to PSL(2, \mathbb{F})$, $\mathbb{F}$ a number field, for which there is an element $\gamma \in \pi_1(M)$ such that $\mathrm{tr}(\rho(\gamma))$ is not an algebraic integer.

(2) There are many closed hyperbolic 3-manifolds M with no essential surfaces. For instance, this is true for any manifold obtained by Dehn surgery on the figure eight knot with surgery coefficient $p/q \in \mathbb{Q} \setminus \{0, \pm1, \pm2, \pm3, \pm4\}$ (see e.g. [HT, Theorem 2(b)]). The preceding example shows that if M is such a manifold and $\mathbb{F}$ is a finite extension of the rationals for which ρ_M conjugates into $PSL(2, \mathbb{F})$, then ρ_M conjugates into $PSL(2, R)$ where R is the ring of integers of $\mathbb{F}$.

Exercise 3.11. Let T be a tree and recall that any two vertices of T are connected by a unique simplicial path without backtracking (i.e., a *geodesic*). Use this fact to complete the following exercises.

(1) Show that the fixed point set $\mathrm{Fix}(\varphi)$ of $\varphi \in \mathrm{Aut}(T)$ is a possibly empty subtree of T.

(2) Suppose that v, w are vertices of T and that $\varphi \in \mathrm{Aut}(T)$ fixes w. Let $[v, w]$ denote the geodesic segment between v and w. Show that there is a point $w' \in [v, w]$ such that

(a) $[w, w'] \subset \text{Fix}(\varphi)$;

(b) $[v, \varphi(v)] = [v, w'] \cup [w', \varphi(v)]$ (see Figure 1).

Conclude that w' is the midpoint of the geodesic segment $[v, \varphi(v)]$ and therefore $\varphi([w', v]) = [w', \varphi(v)]$.

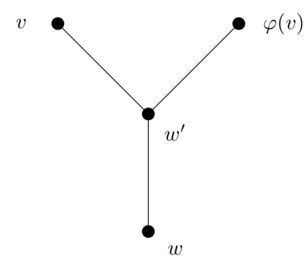

FIGURE 1.

(3) Given disjoint, non-empty subtrees X_1, X_2 of T, show there is a unique geodesic γ of T whose endpoints lie on $X_1 \cup X_2$ such that $\gamma \cap X_1$ and $Y \cap X_2$ are single points.

(4) Suppose that the fixed point sets of $\varphi_1, \varphi_2 \in \text{Aut}(T)$ are non-empty and disjoint. Show that $\varphi_1 \circ \varphi_2$ has no fixed points in T. (Hint: First show that $\text{Fix}(\varphi_1 \circ \varphi_2) \cap \text{Fix}(\varphi_2) = \emptyset$. Next let γ be the geodesic in T connecting $\text{Fix}(\varphi_1)$ and $\text{Fix}(\varphi_2)$, as in part (3) of this exercise, and let $v_1 \in \text{Fix}(\varphi_1), v_2 \in \text{Fix}(\varphi_2)$ be the endpoints of γ. Show that v_1 is the midpoint of the geodesic $[v_2, (\varphi_1 \circ \varphi_2)(v_2)]$, though is not fixed by $\varphi_1 \circ \varphi_2$. Now apply part (2) of the exercise.)

(5) Use induction to show that if $X_1, X_2, \ldots, X_n$ are subtrees of T such that $X_i \cap X_j \neq \emptyset$ for all i, j, then $\bigcap_{i=1}^n X_i \neq \emptyset$. (Hint: Start the induction with the base case $n = 3$.)

(6) Prove that a finitely generated group acts non-trivially on a tree if and only if some element of the group acts fixed point freely on the tree.

4. Algebraic sets of representations

Our goal in this section is to describe how the set of representations of a finitely generated group with values in $SL(2, \mathbb{C})$ can be given the structure of a complex affine algebraic set.

4.1. Representations of 3-manifold groups with values in $SL(2, \mathbb{C})$.

Our understanding of the extent to which the fundamental group of a 3-manifold determines its topology evolved over the last century, culminating with the following result.

Theorem 4.1 (Waldhausen, Mostow, Scott, Perelman). *Let W_1, W_2 be closed, connected, orientable, irreducible 3-manifolds. If $\pi_1(W_1^3) \cong \pi_1(W_2^3)$, then either $W_1 \cong W_2$ or W_1 and W_2 are lens spaces with fundamental group $\mathbb{Z}/p\mathbb{Z}$ for some $p \geq 5$.*

See [AFW, §2.1] for a discussion of this result and how it extends to compact orientable 3-manifolds with boundary.

It is natural, then, to study 3-manifolds via their fundamental groups. As an example of the effectiveness of such an approach, Nathan Dunfield has recently classified a family of over 300,000 closed hyperbolic 3-manifolds by studying the finite quotients of their fundamental groups using machine calculation ([Du, Theorem 1.4]).

Here we study knot manifolds via their representations with values in $SL(2,\mathbb{C})$ and the closely related $PSL(2,\mathbb{C})$. This is a natural choice; as a group of 2×2 matrices, $SL(2,\mathbb{C})$ is simple enough to make effective calculations by machine and even by hand, while it is rich enough to contain many 3-manifold groups. Moreover, the fact that $PSL(2,\mathbb{C})$ is isomorphic to the group of orientation-preserving isometries of hyperbolic 3-space allows us to associate geometric quantities such as volumes to such representations (see [CCGLS, §4.5]).

Examples 4.2.

(1) We saw in Example 3.10 that a complete finite volume orientable hyperbolic 3-manifold admits an essentially unique holonomy representation $\rho_M : \pi_1(M) \to PSL(2,\mathbb{C})$. Thurston used the fact that orientable 3-manifolds have trivial tangent bundles to show that ρ_M lifts to a representation $\pi_1(M) \to SL(2,\mathbb{C})$ (see [CS1, Proposition 3.1.1]). In particular, $\pi_1(M)$ is isomorphic to a subgroup of $SL(2,\mathbb{C})$.

(2) This example uses the theory of 2-orbifolds to construct large families of interesting representations of the fundamental groups of Seifert fibred manifolds. See [Th, Chapter 13] and [Sc, §2, 3] for discussions of 2-orbifolds, their fundamental groups, their Euler characteristics, their geometric structures, and how they arise as the base 2-orbifold of a 3-manifold endowed with the structure of a Sefert fibre space.

Let M be a compact Seifert fibered manifold (i.e., M is foliated by circles) with base orbifold $\mathcal{B} = B(p_1,\dots,p_n)$. There is an exact sequence

$$1 \to \langle h\rangle \to \pi_1(M) \to \pi_1(\mathcal{B}) \to 1$$

where h is the class in $\pi_1(M)$ carried by a regular fibre ([Sc, Lemma 3.2]). For instance, M could be the exterior of the (p,q)-torus knot $(p,q\geq 2)$, in which case $\mathcal{B} = D^2(p,q)$ and the exact sequence is of the form $1 \to \mathbb{Z} \to \pi_1(M) \to \mathbb{Z}/p * \mathbb{Z}/q \to 1$.

In the generic case that $\chi(\mathcal{B}) = \chi(B) - \sum_i (1-\frac{1}{p_i}) < 0$, $\mathcal{B}$ admits a hyperbolic structure for which there is an associated discrete faithful holonomy representation $\pi_1(\mathcal{B}) \to \mathrm{Isom}(\mathbb{H}^2)$ ([Th, Theorem 13.3.6]). It is well-known that $\mathrm{Isom}_+(\mathbb{H}^2) \cong PSL(2,\mathbb{R}) \leq PSL(2,\mathbb{C})$, and we can expand this inclusion to one of all $\mathrm{Isom}(\mathbb{H}^2)$ by extending a glide-reflection along a geodesic in $\mathbb{H}^2$ to a glide-rotation (by π) along the same geodesic considered as lying in $\mathbb{H}^3$. Thus we obtain a discrete faithful representation $\pi_1(\mathcal{B}) \to PSL(2,\mathbb{C})$. The composition

$$\pi_1(M) \to \pi_1(\mathcal{B}) \to \mathrm{Isom}(\mathbb{H}^2) \leq PSL(2,\mathbb{C})$$

yields a representation with kernel $\langle h\rangle$, and the conjugacy classes of all such representations is the *Teichmüller space* of $\mathcal{B} = B(p_1,\dots,p_n)$, which can be topologised to be homeomorphic to $\mathbb{R}^{-3\chi(B)+2n}$ ([Th, Corollary 13.3.7]). The obstruction to lifting this family of representations to $SL(2,\mathbb{C})$ lies in $H^2(M;\mathbb{Z}/2)$ (see [GM, §2]) and may be non-zero, but will necessarily vanish when M is the exterior of a knot in the 3-sphere or, more generally, in a $\mathbb{Z}/2$-homology 3-sphere.

4.2. **Algebraic sets of $SL(2,\mathbb{C})$-representations.** Let $\pi = \langle \gamma_1,\dots,\gamma_n \mid r_i\ (i\in I)\rangle$ be a finitely generated group. Here, we define a naturally arising structure on the set of representations

$$R(\pi) = \{\rho : \pi \to SL(2,\mathbb{C}) \mid \rho \text{ is a homomorphism}\}$$

which potentially encodes deep properties of π. We begin by endowing π with the discrete topology and $R(\pi)$ with the compact-open topology with respect to the the topology on $SL(2,\mathbb{C})$ induced from its natural inclusion into $\mathbb{C}^4$. A clearer picture of this topology is obtained by considering the injective mapping

$$R(\pi) \to SL(2,\mathbb{C})^n, \rho \mapsto (\rho(\gamma_1), \rho(\gamma_2), \ldots, \rho(\gamma_n))$$

Exercise 4.3. Show that the inclusion $R(\pi) \to SL(2,\mathbb{C})^n$ is a topological embedding. That is, show that it induces a homeomorphism between $R(\pi)$ and its image in $SL(2,\mathbb{C})^n \subset \mathbb{C}^{4n}$.

For instance, $R(\mathbb{Z}) \cong SL(2,\mathbb{C})$ and, more generally, if F_n is a free group on n-generators, then $R(F_n) \cong SL(2,\mathbb{C})^n$.

One consequence of the exercise is that $R(\pi)$ is metrisable and if $\{\rho_k\}$ is a sequence in $R(\pi)$, then $\lim_k \rho_k = \rho$ if and only if $\lim_k \rho_k(\gamma_i) = \rho(\gamma_i)$ for $1 \le i \le n$. There is more structure inherent in the embedding $R(\pi) \to SL(2,\mathbb{C})^n$ though, which will appear once we consider its image more carefully.

Each relation r_i is a word in the generators $\gamma_1, \ldots, \gamma_n$ and determines a continuous map

$$r_i : SL(2,\mathbb{C})^n \to SL(2,\mathbb{C}), (A_1, \ldots, A_n) \mapsto r_i(A_1, \ldots, A_n)$$

by formally replacing each appearance of γ_j in r_i by A_j. Thus, if $r_i = \gamma_1^{-1}\gamma_n^3\gamma_2^2$, then $r_i(A_1, \ldots, A_n) = A_1^{-1}A_n^3A_2^2$. From the first isomorphism theorem of groups, we see that an n-tuple $(A_1, \ldots, A_n) \in SL(2,\mathbb{C})^n$ lies in the image of $R(\pi)$ if and only if $r_i(A_1, \ldots, A_n) = I$ for each relation r_i. Thus we have a bijection

$$\begin{array}{rcl} R(\pi) & \equiv & \{(A_1, \ldots, A_n) \in SL(2,\mathbb{C})^n \mid r_i(A_1, \ldots, A_n) = I \text{ for all } i\} \\ \rho & \longmapsto & (\rho(\gamma_1), \rho(\gamma_2), \ldots, \rho(\gamma_n)) \end{array} \tag{4.2.1}$$

If we write $A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$, then as $A_j^{-1} = \begin{pmatrix} d_j & -b_j \\ -c_j & a_j \end{pmatrix}$ we have

$$r_i(A_1, \ldots, A_n) = \begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix}$$

where p_i, q_i, r_i, s_i are integer polynomials in the coordinates a_j, b_j, c_j, d_j $(1 \le j \le n)$. That is,

$$p_i, q_i, r_i, s_i \in \mathbb{Z}[a_j, b_j, c_j, d_j]_{1 \le j \le n}$$

Appealing to (4.2.1), we have

$$R(\pi) \equiv \{(a_j, b_j, c_j, d_j)_{1 \le j \le n} \mid a_jd_j - b_jc_j = 1 \text{ all } j \text{ and } p_i = s_i = 1, q_i = r_i = 0 \text{ all } i\} \subset \mathbb{C}^{4n}$$

Thus, the image in $SL(2,\mathbb{C})^n \subset \mathbb{C}^{4n}$ of $R(\pi)$ is the zero set of a family of polynomials in $4n$ complex variables.

Proposition 4.4. *$R(\pi)$ is naturally identified with a complex affine algebraic set. Further, each $\gamma \in \pi$ determines a regular function*

$$e_\gamma : R(\pi) \to SL(2,\mathbb{C}) \subset \mathbb{C}^4, \rho \mapsto \rho(\gamma)$$

That is, the coordinate functions of e_γ are elements of the coordinate ring $\mathbb{C}[R(\pi)]$ of $R(\pi)$.

Proof. We have already verified the first claim. For the second, write γ as a word w is in the generators $\gamma_1, \gamma_2, \ldots, \gamma_n$ and note that as in our analysis of the relators r_i, there are polynomials $p, q, r, s \in$

$\mathbb{Z}[a_j, b_j, c_j, d_j]_{1 \leq j \leq n}$ such that if $\rho \in R(\pi)$ and $(\rho(\gamma_1), \rho(\gamma_2), \ldots, \rho(\gamma_n)) = (A_1, \ldots, A_n)$, where $A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$, then

$$e_\gamma(\rho) = \rho(\gamma) = w(A_1, \ldots, A_n) = \begin{pmatrix} p(a_*, b_*, c_*, d_*) & q(a_*, b_*, c_*, d_*) \\ r(a_*, b_*, c_*, d_*) & s(a_*, b_*, c_*, d_*) \end{pmatrix}$$

Hence the coordinate functions of e_γ are polynomial in the affine coordinates of $R(\pi)$. □

Ostensibly, our identification of $R(\pi)$ with an algebraic subset of $\mathbb{C}^{4n}$ depends on the choice of generating set of π, but it is a simple consequence of Tietze's theorem on group presentations that the algebraic set is well-defined up to a canonical algebraic isomorphism. We leave the details to the reader.

Exercise 4.5. A representation $\rho \in R(\pi)$ is *reducible* if its image is conjugate into the group of upper-triangular matrices, and *irreducible* otherwise.

(1)(a) Show that $\mathrm{tr}(\rho(\gamma)) = 2$ for each γ in the commutator subgroup $[\pi, \pi]$ of π is a necessary condition for ρ to be reducible.

(b) It is known that the condition that $\mathrm{tr}(\rho(\gamma)) = 2$ for each $\gamma \in [\pi, \pi]$ is sufficient to deduce that ρ is reducible (see [CS1, Corollary 1.2.2]). Use this to show that $R^{red}(\pi) = \{\rho \in R(\pi) \mid \rho \text{ is reducible}\}$ is an algebraic subset of $R(\pi)$.

(2) Show that the set of irreducible representations is open in $R(\pi)$.

Remark 4.6. The material of this section is readily generalised to representations with values in any complex affine algebraic Lie group G. Examples of such G include $PSL(k, \mathbb{C})$ and $SL(k, \mathbb{C})$.

5. Algebraic sets of characters

Throughout this section $\pi = \langle \gamma_1, \ldots, \gamma_n \mid r_i \ (i \in I) \rangle$ will be a finitely generated group. Here we show that the set of characters of the representations in $R(\pi)$ admits the structure of a complex affine algebraic set, as developed in [CS1].

5.1. Characters of $SL(2, \mathbb{C})$-representations.

The *character* of a representation $\rho \in R(\pi)$,

$$\chi_\rho : \pi \to \mathbb{C}, \gamma \mapsto \mathrm{tr}(\rho(\gamma))$$

carries a number of properties of ρ. For instance, ρ is a reducible representation if and only if $\chi_\rho(\gamma) = 2$ for all $\gamma \in [\pi, \pi]$ (see Exercise 4.5).

The invariance of trace under conjugation implies that conjugate representations have the same character. Conversely, if ρ_1 is an irreducible representation and $\chi_{\rho_1} = \chi_{\rho_2}$, then ρ_2 is conjugate to ρ_1. That is, there is some $A \in SL(2, \mathbb{C})$ such that $\rho_2(\gamma) = A\rho_1(\gamma)A^{-1}$ for all $\gamma \in \pi$ (see [CS1, Proposition 1.5.2]). Thus an irreducible representation is determined up to conjugation by its character. It is easy to see that this is false for reducible representations by taking $\pi = \mathbb{Z}$ and $\rho_1, \rho_2 \in R(\mathbb{Z})$ corresponding to $\rho_1(1) = I$ and $\rho_2(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We use $X(\pi)$ to denote the set of characters of the representations contained in $R(\pi)$.

For each $\gamma \in \pi$, consider the evaluation map

$$I_\gamma : X(\pi) \to \mathbb{C}, \chi_\rho \mapsto \chi_\rho(\gamma)$$

The various identities satisfied by the trace of a matrix in $SL(2,\mathbb{C})$ lead to relations between the functions I_γ. For instance, trace is invariant under taking inverse and conjugating, so taking "$\sim$" to denote "is conjugate to", we see that if $\gamma_2 \sim \gamma_1^{\pm 1}$, then $\chi_\rho(\gamma_1) = \chi_\rho(\gamma_2)$ for each $\chi_\rho \in X(\pi)$. Equivalently, $I_{\gamma_1} = I_{\gamma_2}$. Similarly, the identity $\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\mathrm{tr}(B)$ for $A, B \in SL(2,\mathbb{C})$ implies that

$$I_{\gamma_1\gamma_2} + I_{\gamma_1\gamma_2^{-1}} = I_{\gamma_1} I_{\gamma_2} \tag{5.1.1}$$

for all $\gamma_1, \gamma_2 \in \pi$. Taking $\gamma_1 = \gamma_2 = \gamma$ we have

$$I_{\gamma^2} = I_\gamma^2 - 2 \tag{5.1.2}$$

Exercise 5.1.

(1) Use the identity $\mathrm{tr}(A^{-1}) = \mathrm{tr}(A)$ and induction on $|n|$ to show that for each integer n there is a polynomial $q_{|n|}(x)$ with integer coefficients such that if $A \in SL(2,\mathbb{C})$, then $\mathrm{tr}(A^n) = q_{|n|}(\mathrm{tr}(A))$. Thus $I_{\gamma^n} = q_{|n|}(I_\gamma)$ for each $\gamma \in \pi$.

(2) Use (5.1.1) and (5.1.2) to show that if $[\gamma_1, \gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$, then $I_{[\gamma_1,\gamma_2]} = I_{\gamma_1}^2 + I_{\gamma_2}^2 + I_{\gamma_1\gamma_2}^2 - I_{\gamma_1} I_{\gamma_2} I_{\gamma_1\gamma_2} - 2$.

Proposition 5.2. *If π is generated by γ_1, γ_2, then for each $\gamma \in \pi$ there is a polynomial $p_\gamma \in \mathbb{Z}[x, y, z]$ such that for each $\chi_\rho \in X(\pi)$ we have*

$$\chi_\rho(\gamma) = p_\gamma(\chi_\rho(\gamma_1), \chi_\rho(\gamma_2), \chi_\rho(\gamma_1\gamma_2)) = p_\gamma(I_{\gamma_1}(\chi_\rho), I_{\gamma_2}(\chi_\rho), I_{\gamma_1\gamma_2}(\chi_\rho))$$

Hence we have an embedding

$$X(\pi) \to \mathbb{C}^3, \chi_\rho \mapsto (I_{\gamma_1}(\chi_\rho), I_{\gamma_2}(\chi_\rho), I_{\gamma_1\gamma_2}(\chi_\rho))$$

Proof. In the case that $\gamma \sim \gamma_i^a$ for some integer a, the proposition follows from Exercise 5.1. Otherwise, $\gamma \sim \gamma_1^{a_1}\gamma_2^{b_1} \cdots \gamma_1^{a_n}\gamma_2^{b_n}$ for some $n \geq 1$ and non-zero integers a_i, b_i. We complete the proof by induction on the minimal value $N(\gamma)$ of the sums $\sum_i(|a_i| + |b_i|) \geq 2$ over all such representations of a conjugate of γ.

If $N(\gamma) = 2$, then up to taking the inverse of γ, which doesn't affect $\chi(\gamma)$, γ is conjugate to either $\gamma_1\gamma_2$ or $\gamma_1\gamma_2^{-1}$. In the first case we take $p_\gamma(x, y, z) = z$, while in the second we use the identity $\chi(\gamma_1\gamma_2^{-1}) = \chi(\gamma_1)\chi(\gamma_2) - \chi(\gamma_1\gamma_2)$ and take $p_\gamma(x, y, z) = xy - z$.

Assume that $N(\gamma) \geq 3$ and that the proposition holds for each γ' with $N(\gamma') < N(\gamma)$. Fix a conjugate $\gamma_1^{a_1}\gamma_2^{b_1} \cdots \gamma_1^{a_n}\gamma_2^{b_n}$ of γ where $n \geq 1$, the a_i, b_i are non-zero integers, and $N(\gamma) = \sum_i(|a_i| + |b_i|)$.

If $|b_i| \geq 2$ for some i, then up to replacing γ by γ^{-1} and conjugating, we can suppose that $b_n \geq \max\{2, |b_1|, |b_2|, \ldots, |b_{n-1}|\}$. Set $u = \gamma_1^{a_1}\gamma_2^{b_1} \cdots \gamma_1^{a_n}\gamma_2^{b_n-1}$ and note that

$$\chi(\gamma) = \chi(u\gamma_2) = \chi(u)\chi(\gamma_2) - \chi(u\gamma_2^{-1}) \tag{5.1.3}$$

By induction, the proposition holds for u. It also holds $v = u\gamma_2^{-1} = \gamma_1^{a_1}\gamma_2^{b_1} \cdots \gamma_1^{a_n}\gamma_2^{b_n-2}$ since $N(v) < N(\gamma)$. (This is obvious if $b_n \geq 3$ and is readily verified if $b_n = 2$.) Then (5.1.3) shows it holds for γ. A similar argument shows that the proposition holds if $|a_i| \geq 2$ for some i.

Assume then that $|a_i| = |b_i| = 1$ for each i and note that

$$\chi(\gamma) = \chi(uv) = \chi(u)\chi(v) - \chi(uv^{-1}), \tag{5.1.4}$$

where $u = \gamma_1^{a_1}\gamma_2^{b_1}\cdots\gamma_1^{a_{n-1}}\gamma_2^{b_{n-1}}$ and $v = \gamma_1^{a_n}\gamma_2^{b_n}$. By induction, the proposition holds for u and v, and we leave it to the reader to verify that it also holds for

$$uv^{-1} = (\gamma_1^{a_1}\gamma_2^{b_1}\cdots\gamma_1^{a_{n-1}}\gamma_2^{b_{n-1}})(\gamma_2^{-b_n}\gamma_1^{-a_n}) \sim \gamma_1^{a_1-a_n}\gamma_2^{b_1}\cdots\gamma_1^{a_{n-1}}\gamma_2^{b_{n-1}-b_n}$$

Thus it holds for γ by (5.1.4). $\square$

Exercise 5.3. Show that if π is the free group on two generators γ_1, γ_2, the map $X(\pi) \to \mathbb{C}^3, \chi_\rho \mapsto (I_{\gamma_1}(\chi_\rho), I_{\gamma_2}(\chi_\rho), I_{\gamma_1\gamma_2}(\chi_\rho))$ is a bijection. That is if $(x, y, z) \in \mathbb{C}^3$, there is $\rho \in R(\pi)$ such that $(x, y, z) = (I_{\gamma_1}(\chi_\rho), I_{\gamma_2}(\chi_\rho), I_{\gamma_1\gamma_2}(\chi_\rho))$.

The proof of Proposition 5.2 readily generalises to the case $\pi = \langle \gamma_1, \ldots, \gamma_n \mid r_i \ (i \in I)\rangle$ to show that a character $\chi \in X(\pi)$ is determined by its values on the finite set

$$\{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

More precisely, for each $\gamma \in \pi$ there is a polynomial p_γ with integer coefficients such that

$$\chi_\rho(\gamma) = p_\gamma(\chi_\rho(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k})) = p_\gamma(I_{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}}(\chi_\rho))$$

where the subscripts vary over all $i_1, i_2, \ldots, i_k$ such that $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. Consequently, we obtain an embedding

$$X(\pi) \to \mathbb{C}^{2^n-1}, \chi \mapsto (I_{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}}(\chi))$$

Unlike the case $n = 2$, the map is not onto for $n \geq 3$, though its image is always closed ([CS1, Corollary 1.4.5]).

5.2. **Algebraic sets of $SL(2,\mathbb{C})$-characters.** Next we show that $X(\pi)$ inherits the structure of a complex affine algebraic set from $R(\pi)$ via the map

$$t : R(\pi) \to X(\pi), \rho \mapsto \chi_\rho$$

To see this, recall from the proof of Proposition 4.4 that for $\gamma \in \pi$, the coordinate functions of the evaluation map $e_\gamma : R(\pi) \to SL(2,\mathbb{C}) \subset \mathbb{C}^4, \rho \mapsto \rho(\gamma)$, lie in the coordinate ring $\mathbb{C}[R(\pi)]$ of $R(\pi)$. Hence the composition $\mathrm{tr} \circ e_\gamma$ lies in $\mathbb{C}[R(\pi)]$. On the other hand,

$$\mathrm{tr}(e_\gamma(\rho)) = \chi_\rho(\gamma) = I_\gamma(t(\rho))$$

It follows that the composition

$$R(\pi) \xrightarrow{t} X(\pi) \subseteq \mathbb{C}^N, \rho \mapsto \chi_\rho \equiv (I_{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}}(\chi_\rho))$$

is a regular function whose image is closed in $\mathbb{C}^N$ ([CS1, Corollary 1.4.5]). A fundamental theorem in algebraic geometry now implies that this image is a $\mathbb{C}$-affine algebraic subset of $\mathbb{C}^N$ (see [Shf, Theorem 2, page 45]). Of course the image depends on the set of generators chosen for π, but as in the case of $R(\pi)$, the algebraic set it determines is well-defined up to a canonical isomorphism of algebraic sets.

Proposition 5.4. *$X(\pi)$ admits the structure of a $\mathbb{C}$-affine algebraic set in such a way that $t : R(\pi) \to X(\pi), \rho \mapsto \chi_\rho$ is a regular function. Further, $I_\gamma \in \mathbb{C}[X(\pi)]$ for each $\gamma \in \pi$.* $\square$

The $SL(2,\mathbb{C})$-*character variety* of π is $X(\pi)$ endowed with this structure. For instance, the character variety of the free group on two generators is the affine space $\mathbb{C}^3$ (Exercise 5.3).

Example 5.5. Consider the case that π is the two-generator group $\mathbb{Z}/p * \mathbb{Z}/q = \langle \gamma_1, \gamma_2 \mid \gamma_1^p, \gamma_2^q \rangle$, where $p, q \geq 2$. Proposition 5.2 shows that the map $\chi_\rho \mapsto (I_{\gamma_1}(\chi_\rho), I_{\gamma_2}(\chi_\rho), I_{\gamma_1\gamma_2}(\chi_\rho))$ embeds $X(\pi)$ into $\mathbb{C}^3$. Since the trace of any $A \in SL(2,\mathbb{C})$ of order $n \geq 1$ equals $\mathrm{tr}(A) = 2\cos(2\pi j/n)$ for some $0 \leq j \leq n-1$, the image of $X(\pi)$ in $\mathbb{C}^3$ is contained in

$$\bigcup_{j=0}^{p-1} \bigcup_{k=0}^{q-1} \{(2\cos(2\pi j/p), 2\cos(2\pi k/q))\} \times \mathbb{C}$$

and hence is a finite union of isolated points and affine lines.

Exercises 5.6.

(1) Determine the image of $X(\mathbb{Z}/p * \mathbb{Z}/q)$ in $\mathbb{C}^3$ under the map $\chi_\rho \mapsto (I_{\gamma_1}(\chi_\rho), I_{\gamma_2}(\chi_\rho), I_{\gamma_1\gamma_2}(\chi_\rho))$.

(2)(a) Show that the set $X^{red}(\pi)$ of reducible characters of representations $\rho \in R(\pi)$ is a closed algebraic subset of $X(\pi)$ (see Exercise 4.5).

(b) Show that if K is a knot in S^3 and M_K is its exterior, then $X^{red}(M_K) \cong \mathbb{C}$. (Hint: Use the fact that $X^{red}(M_K) = \{\chi_\rho \in X(M_K) \mid \rho \text{ has diagonal image}\}$.)

Define $X^{irr}(\pi)$ to be the algebraic subset of $X(\pi)$ given by

$$X^{irr}(\pi) = \overline{X(\pi) \setminus X^{red}(\pi)} \subseteq X(\pi)$$

An algebraic set X admits a unique decomposition as the union of finitely many distinct irreducible algebraic subsets $X = X_1 \cup X_2 \ldots \cup X_n$, where by *irreducible* we mean that any time we express X_i as a union $X_i' \cup X_i''$ of algebraic subsets, either $X_i = X_i'$ or $X_i = X_i''$. The X_i are called the *algebraic components* of X, and we note that for each $i \neq j$, $X_i \cap X_j$ is a proper algebraic subset of both X_i and X_j, hence nowhere dense in each.

The example and exercises which follow determine the algebraic components of the $SL(2,\mathbb{C})$-character varieties of the exteriors of torus knots and the figure eight knot.

Example 5.7. Let $T(2,3)$ be the trefoil knot and M its exterior. We use $R(M)$ and $X(M)$ to denote, respectively, $R(\pi_1(M))$ and $X(\pi_1(M))$.

Exercise 5.6(2)(b) shows that $X^{red}(M) \cong \mathbb{C}$. We claim that $X^{irr}(M) \cong \mathbb{C}$ as well. To see this, recall

$$\pi_1(M) = \langle \gamma_1, \gamma_2 \mid \gamma_1^2 = \gamma_2^3 \rangle$$

where $\gamma_1^2 = \gamma_2^3$ is central in $\pi_1(M)$. If $\rho \in R(M)$ is irreducible, it is a simple exercise in matrix multiplication to see that the only possibilities for the central elements of the image of ρ are I and $-I$. Then $\rho(\gamma_1)^2 = \rho(\gamma_2)^3 \in \{\pm I\}$, though $\rho(\gamma_1) \neq \pm I$ as otherwise ρ is reducible. Hence $\rho(\gamma_1)$ has order 4 and is therefore conjugate to $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. In particular,

$$\chi_\rho(\gamma_1) = 0$$

Further, $\rho(\gamma_2)^3 = \rho(\gamma_1)^2 = -I$, so $\rho(\gamma_2)$ has order 6, which implies $\rho(\gamma_2) \sim \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and therefore

$$\chi_\rho(\gamma_2) = 1$$

Then the image of $X^{irr}(M)$ under the embedding $X(M) \to \mathbb{C}^3, \chi \mapsto (\chi(\gamma_1), \chi(\gamma_2), \chi(\gamma_1\gamma_2))$ of Proposition 5.2 is contained in the irreducible curve $X_1 = \{(0, 1, w) \mid w \in \mathbb{C}\} \cong \mathbb{C}$. On the other hand, for

$z \in \mathbb{C}$ if $\rho_z : \pi_1(M) \to SL(2, \mathbb{C})$ is defined by

$$\rho_z(\gamma_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \rho_z(\gamma_2) = \begin{pmatrix} z & -(z^2 - z + 1) \\ 1 & 1 - z \end{pmatrix}$$

then $\chi_{\rho_z}(\gamma_1\gamma_2) = 2iz - i$, so each $(0, 1, w) \in \{(0, 1)\} \times \mathbb{C}$ corresponds to a character of $\pi_1(M)$. Thus the algebraic decomposition of $X(M)$ is

$$X(M) = X_0 \cup X_1$$

where $X_0 = X^{irr}(M) \equiv \{(0, 1, w) \mid w \in \mathbb{C}\} \cong \mathbb{C}$ and $X_1 = X^{red}(M) \cong \mathbb{C}$.

Exercises 5.8.

(1) Let $K = T(p, q)$ be the (p, q)-torus knot where $2 \leq p \leq q$. It is known that

$$\pi_1(M_K) \cong \langle \gamma_1, \gamma_2 \mid \gamma_1^p = \gamma_2^q \rangle$$

Determine the image of the embedding $X(M_K) \to \mathbb{C}^3, \chi \mapsto (\chi(\gamma_1), \chi(\gamma_2), \chi(\gamma_1\gamma_2))$ and consequently the algebraic decomposition of $X(M_K)$.

(2) Let K be the figure eight knot. The goal of this problem is to use an explicit embedding $X(M_K) \to \mathbb{C}^2$ to determine the algebraic decomposition of $X(M_K)$.

Recall that

$$\pi_1(M_K) = \langle \gamma_1, \gamma_2 \mid w\gamma_1 = \gamma_2 w \rangle$$

where w is the commutator $[\gamma_1^{-1}, \gamma_2] = \gamma_1^{-1}\gamma_2\gamma_1\gamma_2^{-1}$ of γ_1^{-1} and γ_2. As $\pi_1(M_K)$ is generated by two elements, $X(M_K)$ embeds in $\mathbb{C}^3$ via $\chi_\rho \mapsto (\chi_\rho(\gamma_1), \chi_\rho(\gamma_2), \chi_\rho(\gamma_1\gamma_2^{-1}))$ (Proposition 5.2), and as $\gamma_2 = w\gamma_1 w^{-1}$, we have $\chi_\rho(\gamma_2) = \chi_\rho(\gamma_1)$ for each ρ. Thus, $X(M_K)$ embeds in $\mathbb{C}^2$ via $\chi_\rho \mapsto (\chi_\rho(\gamma_1), \chi_\rho(\gamma_1\gamma_2^{-1}))$.

(a) Show that the image of $X^{red}(M)$ in $\mathbb{C}^2$ under the map $\chi \mapsto (\chi_\rho(\gamma_1), \chi_\rho(\gamma_1\gamma_2^{-1}))$ is $\mathbb{C} \times \{2\}$.

(b) Suppose that ρ is irreducible and set

$$A = \rho(\gamma_1), \quad B = \rho(\gamma_2), \quad W = \rho(w)$$

(i) Show that up to replacing ρ with a conjugate representation we can assume that

$$A = \begin{pmatrix} a & 1 \\ 0 & a^{-1} \end{pmatrix} \quad B = \begin{pmatrix} a & 0 \\ s & a^{-1} \end{pmatrix}$$

for some $a \in \mathbb{C}^*$ and $s \neq 0$. Set

$$x = \mathrm{tr}(A) = a + a^{-1}, \quad y = \mathrm{tr}(AB^{-1}) = 2 - s$$

(ii) Show that $WA = BW$ if and only if $y^2 - x^2y + x^2 + y - 1 = 0$ and conclude that

$$X^{irr}(M_K) = \{(x, y) \mid y^2 - x^2y + x^2 + y - 1 = 0\}$$

(Hint: This is an elementary, though tedious, calculation. The idea is to write WA and BW as matrices whose coefficients are functions in the variables a and s. Comparing coefficients shows that $WA = BW$ if and only if their $(1, 2)$ entries are equal, and the latter occurs if and only if $y^2 - x^2y + x^2 + y - 1 = 0$.)

Thus $X(M) = X_0 \cup X_1$ where $X_0 = X^{irr}(M) \cong \{(x, y) \mid y^2 - x^2y + x^2 + y - 1 = 0\}$, which can be shown to be an irreducible curve of genus 1, and $X_1 = X^{red}(M) \cong \mathbb{C}$.

Remarks 5.9.

(1) The method of analysis used in Exercise 5.8(2) can be applied to study the character variety of a general 2-bridge knot $K_{p,q}$, where p, q are odd integers satisfying $-q < p < q$. (The figure eight knot is $K_{3,5}$.) In fact,

$$\pi_1(M_{K_{p,q}}) \cong \langle \gamma_1, \gamma_2 \mid w\gamma_1 = \gamma_2 w \rangle$$

where

$$w = \gamma_1^{\epsilon_1}\gamma_2^{\epsilon_2}\gamma_1^{\epsilon_3}\gamma_2^{\epsilon_4}\cdots\gamma_1^{\epsilon_{q-2}}\gamma_2^{\epsilon_{q-1}}$$

and $\epsilon_i = (-1)^{\lfloor \frac{i|p|}{q} \rfloor}$. The reader may wish to work out several examples for small values of p, q.

(2) We saw above that the character variety of a free group on two generators was the affine space $\mathbb{C}^3$. In contrast to this, the character varieties of free groups on three or more generators are never affine coordinate spaces; certain "trace identity" polynomials are needed to define them. We refer the reader to [GM] for an alternate development of the material of this section which makes explicit the polynomials defining $X(\pi)$ which arise from "trace identities" and those which arise from group relations.

6. Ideal points and discrete valuations

Throughout this section π will be a finitely generated group, say $\pi = \langle \gamma_1, \ldots, \gamma_n \mid r_i \ (i \in I)\rangle$, and $X_0 \subset X^{irr}(\pi)$ will be an irreducible curve. The irreducibility of X_0 implies that its coordinate ring $\mathbb{C}[X_0]$ is a domain. Its field of fractions $\mathbb{C}(X_0)$ is called the *function field* of X_0.

Our goal is to show the ideal points of X_0 give rise to non-trivial actions of π on trees. This material is due to Culler and Shalen [CS1].

6.1. **Smooth models and discrete valuations.** Complex affine algebraic varieties are smooth manifolds in the complement of a proper algebraic subset, so X_0 is a surface in the complement of a finite subset. Section 1.5.3 of [CGLS] discusses the classic result that there is a projective variety $\widetilde{X}_0$, an affine variety X_0^ν, and regular maps

$$\widetilde{X}_0 \supset X_0^\nu \xrightarrow{\nu} X_0$$

where

- X_0^ν is a smooth $\mathbb{C}$-affine set and $X_0^\nu \xrightarrow{\nu} X_0$ is a desingularisation. (So away from a finite subset of X_0, ν is an isomorphism.) The elements of X_0^ν are called the *ordinary points* of $\widetilde{X}_0$;
- $\widetilde{X}_0$ is a smooth $\mathbb{C}$-projective set (and therefore a compact Riemann surface) containing X_0^ν, where $\widetilde{X}_0 \setminus X_0^\nu$ is a finite set whose elements are called the *ideal points* of $\widetilde{X}_0$.

A key point for us is that there are isomorphisms

$$\mathbb{C}(\widetilde{X}_0) \xrightarrow{f \mapsto f|_{X_0^\nu}} \mathbb{C}(X_0^\nu) \xleftarrow{f\circ\nu \leftarrow\!\shortmid f} \mathbb{C}(X_0)$$

which we use to identify the function fields of X_0, X_0^ν, and $\widetilde{X}_0$. In what follows we will write $\mathbb{C}(X_0)$ for each of these fields.

We can use the Riemann surface structure on $\widetilde{X}_0$ to define a discrete valuation v_x on $\mathbb{C}(X_0)$ for each point x of $\widetilde{X}_0$, where

$$v_x(f) = n \quad \text{if} \quad \begin{cases} f(x) \in \mathbb{C} \text{ and } f \text{ has a zero of order } n \geq 0 \text{ at } x \\ \\ f(x) = \infty \text{ and } f \text{ has a pole of order } -n > 0 \text{ at } x \end{cases}$$

(see Example 3.2). The valuation ring $\mathcal{O}(v_x)$ of v_x is the ring of meromorphic functions on $\widetilde{X}_0$ which are holomorphic at x.

6.2. **The tautological representation.** The inverse image $t^{-1}(X_0) \subset R(\pi)$ is an algebraic set and hence the union of finitely many irreducible components. We can argue as in the proof of [BZ2, Lemma 4.1] to see that exactly one of these components, R_0 say, satisfies $t(R_0) = X_0$. Further, R_0 is closed under conjugation and has complex dimension 4. The latter follows from the former since X_0 has dimension 1 and the inverse image of an irreducible character on X_0 is the $SL(2,\mathbb{C})$-conjugacy class of an irreducible representation, which is homeomorphic to the 3-dimensional space $PSL(2,\mathbb{C})$.

Since $t|_{R_0} : R_0 \to X_0$ is regular and surjective, we have an inclusion of fields

$$\mathbb{C}(X_0) \to \mathbb{C}(R_0), f \mapsto f \circ t|_{R_0}$$

In other words, for $f \in \mathbb{C}(X_0)$ and $\rho \in R_0$ we set

$$f(\rho) := f(\chi_\rho) \tag{6.2.1}$$

Let $x \in \widetilde{X}_0$ and v_x be the associated valuation. A well-known result in commutative algebra guarantees the existence of an integer $e \geq 1$ and a valuation ω_x on $\mathbb{C}(R_0)$ such that $\omega_x|_{\mathbb{C}(X_0)} = ev_x$ (see [CGLS, Theorem 1.2.3]). Then

$$\mathcal{O}(\omega_x) \cap \mathbb{C}(X_0) = \mathcal{O}(v_x)$$

Let $T(\omega_x)$ be the tree of ω_x (§3.2) and recall that $SL(2,\mathbb{C}(R_0))$ acts simplicially on $T(\omega_x)$ without inversions (§3.3).

Next recall that for each $\gamma \in \pi$, the functions $a_\gamma, b_\gamma, c_\gamma, d_\gamma : R_0 \to \mathbb{C}$ determined by the evaluation map

$$e_\gamma : R_0 \to SL(2,\mathbb{C}) \subset \mathbb{C}^4, \rho \mapsto \rho(\gamma) = \begin{pmatrix} a_\gamma(\rho) & b_\gamma(\rho) \\ c_\gamma(\rho) & d_\gamma(\rho) \end{pmatrix}$$

lie in $\mathbb{C}[R_0]$ (Proposition 4.4). It is readily seen that the map $P : \pi \to SL(2,\mathbb{C}(R_0))$ defined by

$$P(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$$

is a homomorphism, and we call P the *tautological representation* of π in $SL(2,\mathbb{C}(R_0))$. Under our embedding of $\mathbb{C}(X_0)$ in $\mathbb{C}(R_0)$ (see Equation (6.2.1)), we have $\mathrm{tr}(P(\gamma)(\rho)) = \mathrm{tr}(\rho(\gamma)) = I_\gamma(\chi_\rho)$, so that

$$\mathrm{tr}(P(\gamma)) = I_\gamma$$

Consider the action of π on $T(\omega_x)$ determined by P.

Lemma 6.1. *$\gamma \in \pi$ fixes a vertex of $T(\omega_x)$ if and only if $I_\gamma(x) \in \mathbb{C}$.*

Proof. Proposition 3.4(1) shows that $\gamma \in \pi_1(M)$ fixes a vertex of $T(\omega_x)$ if and only if $\mathrm{tr}(P(\gamma)) \in \mathcal{O}(\omega_x)$. On the other hand, we saw above that $\mathrm{tr}(P(\gamma)) : R_0 \to \mathbb{C}$ equals $I_\gamma \in \mathbb{C}(X_0) \subset \mathbb{C}(R_0)$, so γ fixes a vertex of $T(\omega_x)$ if and only if $I_\gamma \in \mathcal{O}(\omega_x)$. But by construction, $\omega_x(I_\gamma) = ev_x(I_\gamma)$, so $I_\gamma \in \mathcal{O}(\omega_x)$ if and only if $I_\gamma \in \mathcal{O}(v_x)$. That is, if and only if $I_\gamma(x) \in \mathbb{C}$. □

Proposition 6.2. *The action of π on $T(\omega_x)$ is non-trivial if and only if x is an ideal point.*

Proof. If x is ordinary, $I_\gamma(x) \in \mathbb{C}$ for all $\gamma \in \pi_1(M)$. Thus each element of π fixes a vertex of $T(\omega_x)$ by Lemma 6.1, so π fixes a vertex of $T(\omega_x)$ by Exercise 3.11(6).

If x is ideal, there is some $\gamma \in \pi$ such that $I_\gamma(x) = \infty$. (This follows from [CS1, Corollary 1.4.5].) Then by Lemma 6.1, γ has no fixed point in $T(\omega_x)$, so the action is non-trivial. $\square$

Remarks 6.3.

(1) Referring to Remark 2.7, Proposition 6.2 shows that if π admits no non-trivial splitting as a free product with amalgamation or an HNN extension, then $X(\pi)$ is 0-dimensional and hence a finite set. For instance, the remark shows that this is the case if π is the fundamental group of a 3-manifold without any essential surfaces.

(2) Gromov-Hausdorff convergence theory provides a picture of how an ideal point of a curve X_0 of $SL(2,\mathbb{C})$-characters of π leads to an action on a tree. First note that as $\mathrm{Isom}_+(\mathbb{H}^3) \cong PSL(2,\mathbb{C})$, any representation $\rho : \pi \to SL(2,\mathbb{C})$ determines an action of π on $\mathbb{H}^3$. Hence if $\{\chi_{\rho_n}\}$ is a sequence of characters in X_0 which converge to an ideal point, we obtain a sequence of such actions using the ρ_n. By hypothesis, the values $\{\mathrm{tr}(\rho_n(\gamma))\}$ blow up for some element $\gamma \in \pi$. Proceeding with care, we can replace each ρ_n by a conjugate representation and rescale the metric on $\mathbb{H}^3$ so that a subsequence of these actions converge in the Gromov-Hausdorff sense to a complete metric space T on which π acts by isometries. Moreover, the curvature of T is the limit of the curvatures of the rescaled versions of $\mathbb{H}^3$. In other words, T has negatively infinite curvature, like trees do. In fact, the hypothesis that the χ_n are constrained to lie on a curve of $SL(2,\mathbb{C})$-characters can be used to show that T is a simplicial tree and the action is simplicial. Even without this constraint on the χ_n, we can still conclude that T is an $\mathbb{R}$-tree, and so π is subject to the powerful results concerning groups acting by isometries on $\mathbb{R}$-trees. See [Be], [Co], or [Pa] for more detailed discussions, and [Ti] for a related approach due to Thurston.

7. Culler-Shalen seminorms

In this section we describe Culler and Shalen's quantification, detailed in [CGLS, Chapter 1], of the work in the previous section.

7.1. **Poles and ideal points.** Let M be a knot manifold and $X_0 \subset X^{irr}(M)$ an irreducible curve. As in the previous section, R_0 will denote the 4-dimensional subvariety of $R(\pi)$ uniquely determined by the condition that $t(R_0) = X_0$.

Let x be an ideal point of $\widetilde{X}_0$, υ_x the associated discrete valuation and ω_x a valuation on the function field $\mathbb{C}(R_0) \supset \mathbb{C}(X_0)$ satisfying $\omega_x|_{\mathbb{C}(X_0)} = e\upsilon_x$ for some integer $e \geq 1$ (see §6.2).

For $\gamma \in \pi_1(M)$ define $f_\gamma \in \mathbb{C}[X_0] \subset \mathbb{C}(\widetilde{X}_0)$ by

$$f_\gamma(\chi) = I_\gamma(\chi)^2 - 4$$

and denote the order of x as a pole of f_γ by $\Pi_x(f_\gamma) \geq 0$. Here is the key lemma for the construction of Culler-Shalen seminorms.

Lemma 7.1. *For each ideal point x of $\widetilde{X}_0$, there is a homomorphism $\phi_x : \pi_1(\partial M) = H_1(\partial M) \to \mathbb{Z}$ such that*

$$|\phi_x(\gamma)| = \Pi_x(f_\gamma)$$

for all $\gamma \in \pi_1(\partial M)$.

Proof. Recall the tautological representation $P : \pi \to SL(2, \mathbb{C}(R_0))$ defined in §6.2.

First suppose that the eigenvalues of $P(\alpha)$ lie in $\{\pm 1\}$ for each $\alpha \in \pi_1(\partial M)$. Then for such α we have $f_\alpha = I_\alpha^2 - 4 = \mathrm{tr}(P(\alpha))^2 - 4 = 0$ and therefore $\Pi_x(f_\alpha) = 0$. Hence we take $\phi_x = 0$.

Next suppose that there is some $\gamma \in \pi_1(\partial M)$ such that the eigenvalues of $P(\gamma)$ are not contained in $\{\pm 1\}$. Then there is a generating set $\{\alpha, \beta\}$ of $\pi_1(\partial M)$ such that the eigenvalues of $P(\alpha)$ are not contained in $\{\pm 1\}$.

If the eigenvalues of $P(\alpha)$ lie in $\mathbb{C}(R_0)$, we can conjugate the representation P in $SL(2, \mathbb{C}(R_0))$ to obtain a new representation $P' : \pi \to SL(2, \mathbb{C}(R_0))$ such that $P'(\alpha)$ is a diagonal matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \neq \pm I$. Then as $P'(\alpha)$ and $P'(\beta)$ commute, $P'(\beta)$ is also diagonal, say $P'(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$. Then for integers n, m,

$$f_{\alpha^n\beta^m} = \mathrm{tr}(P'(\alpha^n\beta^m))^2 - 4 = \mathrm{tr}\big(\begin{pmatrix} \lambda_1^n\lambda_2^m & 0 \\ 0 & \lambda_1^{-n}\lambda_2^{-m} \end{pmatrix}\big)^2 - 4 = (\lambda_1^n\lambda_2^m - \lambda_1^{-n}\lambda_2^{-m})^2$$

We claim that

$$\Pi_x(f_{\alpha^n\beta^m}) = (2/e)|\omega_x(\lambda_1)n + \omega_x(\lambda_2)m|$$

This is certainly true when $\lambda_1^n\lambda_2^m = \pm 1$, as in this case $f_{\alpha^n\beta^m} = 0$ and therefore

$$\Pi_x(f_{\alpha^n\beta^m}) = 0 = (2/e)|\omega_x(\lambda_1^n\lambda_2^m)| = (2/e)|\omega_x(\lambda_1)n + \omega_x(\lambda_2)m|$$

Suppose then that $\lambda_1^n\lambda_2^m \neq \pm 1$ and observe that from the definition of υ_x we have

$$\Pi_x(f_{\alpha^n\beta^m}) = -\min\{0, \upsilon_x(f_{\alpha^n\beta^m})\} = -(1/e)\min\{0, \omega_x(f_{\alpha^n\beta^m})\}$$

Then

$$\Pi_x(f_{\alpha^n\beta^m}) = -(1/e)\min\{0, \omega_x((\lambda_1^n\lambda_2^m - \lambda_1^{-n}\lambda_2^{-m})^2)\} = -(2/e)\min\{0, \omega_x(\lambda_1^n\lambda_2^m - \lambda_1^{-n}\lambda_2^{-m})\}$$

Exercise 7.2 (below) now shows that

$$\Pi_x(f_{\alpha^n\beta^m}) = (2/e)|\omega_x(\lambda_1^n\lambda_2^m)| = (2/e)|\omega_x(\lambda_1)n + \omega_x(\lambda_2)m|$$

Taking $\phi_x(\alpha^n\beta^m)$ to be $(2/e)(\omega_x(\lambda_1)n + \omega_x(\lambda_2)m)$ completes the proof when the eigenvalues of $P(\alpha)$ lie in $\mathbb{C}(R_0)$.

In the final case that the eigenvalues of $P(\alpha)$ are not contained in $\mathbb{C}(R_0)$, pass to the degree-two extension $\mathbb{F}$ of $\mathbb{C}(R_0)$ which contains them and choose a discrete valuation ω on $\mathbb{F}$ such that $\omega|_{\mathbb{C}(R_0)} = e'\omega_x$ for some integer $e' \geq 1$. Now repeat the argument of the previous paragraph with $\mathbb{F}$ replacing $\mathbb{C}(R_0)$ and ω replacing ω_x. □

Exercise 7.2. Show that if υ is a discrete valuation on a field $\mathbb{F}$ and $y \in \mathbb{F} \setminus \{-1, 0, 1\}$, then

$$|\upsilon(y)| = -\min\{0, \upsilon(y - y^{-1})\}$$

Definition 7.3. A *boundary slope* of M is a slope α on ∂M for which there is an essential surface F in M whose boundary is a non-empty union of curves of slope α.

Lemma 7.4. *Let x be an ideal point of $\widetilde{X}_0$ and $\phi_x : \pi_1(\partial M) = H_1(\partial M) \to \mathbb{Z}$ the homomorphism for which $|\phi_x(\gamma)| = \Pi_x(f_\gamma)$. Then,*

(1) *If $\phi_x \equiv 0$, there is a non-empty closed essential surface in M dual to the action of $\pi_1(M)$ on $T(\omega_x)$;*

(2) *If $\phi_x \not\equiv 0$, let $\alpha \in \pi_1(\partial M)$ generate its kernel. Then any essential surface in M dual to the action of $\pi_1(M)$ on $T(\omega_x)$ has non-empty boundary of slope α. Hence α is a boundary slope.*

Proof. The action of $\pi_1(M)$ on $T(\omega_x)$ is non-trivial since x is ideal (Lemma 6.2). Hence no dual surface to this action is empty (§2.2).

Next note that if $\gamma \in \pi_1(\partial M)$ and $\phi_x(\gamma) = 0$, then f_γ does not have a pole at x and therefore $I_\gamma(x)^2 - 4 = f_\gamma(x) \in \mathbb{C}$. Lemma 6.1 then shows that γ fixes a vertex of $T(\omega_x)$. Hence if $\phi_x \equiv 0$, each element of $\pi_1(\partial M)$ fixes some vertex of $T(\omega_x)$ and therefore $\pi_1(\partial M)$ fixes a vertex of $T(\omega_x)$ by Exercise 3.11(6). We can assume, then, that in the construction of the equivariant map $f : \widetilde{M} \to T(\omega_x)$ in Proposition 2.3, each component of $\partial\widetilde{M}$ is sent to a vertex of $T(\omega_x)$. This implies that the associated dual surface S is contained in the interior of M and is therefore closed. Now modify S, as in the proof of Theorem 2.6, to produce a closed essential surface dual to the action of $\pi_1(M)$ on $T(\omega_x)$. This proves (1).

Next suppose that $\phi_x \not\equiv 0$ and $\alpha \in \pi_1(\partial M)$ generates its kernel. We can assume that the 1-skeleton of the restriction of the triangulation of M to ∂M used in the proof of Proposition 2.3 contains a simple closed curve C of slope α and a simple closed curve C' of slope $\beta \neq \alpha$.

Since $\Pi_x(f_\beta) = |\phi_x(\beta)| > 0$, $I_\beta(x) = \infty$ and therefore β has no fixed vertices in $T(\omega_x)$ (Lemma 6.1). Then if $\widetilde{C'}$ is a component of the inverse image of C' in $\partial\widetilde{M}$ and $f : \widetilde{M} \to T(\omega_x)$ is any equivariant map constructed in Proposition 2.3, $f(\widetilde{C'}) \cap E \neq \emptyset$. Hence $C' \cap S \neq \emptyset$, from which we conclude that any surface in M dual to the action of $\pi_1(M)$ on $T(\omega_x)$ has non-empty boundary.

On the other hand, $\phi_x(\alpha) = 0$, so α has a fixed point in $T(\omega_x)$. We can assume that in constructing $f : \widetilde{M} \to T(\omega_x)$, each component of the inverse image of C in $\widetilde{M}$ is sent to a vertex of $T(\omega_x)$. Then if S is the associated dual surface to the action, $C \cap \partial S = \emptyset$. It follows that any essential dual surface constructed as in the proof of Theorem 2.6 has non-empty boundary disjoint from C. Thus it has slope α, which proves (2). $\square$

7.2. **Culler-Shalen seminorms.**

Exercise 7.5. Let $V \cong \mathbb{R}^2$ and suppose that for $1 \leq j \leq n$ we have a non-zero homomorphism $\phi_j : V \to \mathbb{R}$. Define

$$\| \cdot \| : V \to [0, \infty)$$
$$v \mapsto \sum_j |\phi_j(v)|$$

(a) Show that $\| \cdot \|$ is a (non-zero) seminorm.

(b) If $\| \cdot \|$ is a norm, show that its ball B of radius 1 is a finite-sided balanced[1] polygon, each of whose vertices lie on one of the lines $\ker(\phi_j)$. Further, for each j there is a vertex of B lying on the line $\ker(\phi_j)$.

(c) If $\| \cdot \|$ is not a norm, determine its ball of radius 1.

[1] B is *balanced* if it is invariant under multiplication by -1.

For each ideal point x of X_0, extend ϕ_x to a linear transformation $H_1(\partial M;\mathbb{R}) \to \mathbb{R}$ which, with a mild abuse of notation, we continue to denote by ϕ_x.

Define the *Culler-Shalen seminorm* of X_0 to be the seminorm $\|\cdot\|_{X_0} : H_1(\partial M;\mathbb{R}) \to \mathbb{R}$ given by

$$\|v\|_{X_0} = \sum_x |\phi_x(v)|$$

For $\gamma \in \pi_1(\partial M)$ we have

$$\|\gamma\|_{X_0} = \sum_{x \text{ ideal}} \Pi_x(f_\gamma) = \sum_{x \in \widetilde{X}_0} \Pi_x(f_\gamma) = \text{degree}(f_\gamma : \widetilde{X}_0 \to \mathbb{C}P^1) \tag{7.2.1}$$

(Recall that the degree of a holomorphic map $\widetilde{X}_0 \to \mathbb{C}P^1$ is the sum of the orders of its poles.) There are three possibilities for $\|\cdot\|_{X_0}$: it is either a norm, or a non-zero seminorm which is not a norm, or identically zero, and each possibility can arise.

Exercise 7.6. Use Lemma 7.4 to show that if $\|\cdot\|_{X_0}$ is not a norm, there is a boundary slope β and an integer $s \geq 0$ such that for all $\alpha \in H_1(\partial M)$,

$$\|\alpha\|_{X_0} = s|\alpha \cdot \beta|,$$

where $|\alpha \cdot \beta|$ is the absolute value of the algebraic intersection of α and β.

Example 7.7. Let M be the trefoil exterior so that $\pi_1(M) = \langle \gamma_1, \gamma_2 \mid \gamma_1^2 = \gamma_2^3 \rangle$. The meridional class is given by $\mu = \gamma_1\gamma_2^{-1}$ and the class of a regular fibre by $h = \gamma_1^2$. It was shown in Example 5.7 that $X^{irr}(M)$ is an irreducible curve $X_0 \cong \mathbb{C}$ which consists of the characters of the representations $\rho_z : \pi_1(M) \to SL(2,\mathbb{C})$ given by

$$\rho_z(\gamma_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \rho_z(\gamma_2) = \begin{pmatrix} z & -(z^2 - z + 1) \\ 1 & 1 - z \end{pmatrix}$$

were $z \in \mathbb{C}$. Then $\rho_z(h) = \rho_z(\gamma_1)^2 = -I$ for all z so that $f_h(\chi_{\rho_z}) = 0$ for all z. Identity (7.2.1) now shows

$$\|h\|_{X_0} = \text{degree}(f_h : X_0 \to \mathbb{C}) = 0$$

By Exercise 7.6 there is an integer $s \geq 0$ such that $\|\alpha\|_{X_0} = s\Delta(\alpha, h)$. Since $\Delta(\mu, h) = 1$, we have $s = \|\mu\|_{X_0}$. To calculate $\|\mu\|_{X_0}$, the identity

$$f_\mu(\chi_{\rho_z}) = \text{tr}(\rho_z(\gamma_1\gamma_2^{-1}))^2 - 4 = -(4z^2 - 4z + 5)$$

implies that f_μ has a pole of order 2 at the one ideal point of X_0. It follows that $s = \|\mu\|_{X_0} = \text{degree}(f_\mu|X_0) = 2$ (see (7.2.1)). Hence

$$\|\alpha\|_{X_0} = 2|\alpha \cdot h|$$

Exercise 7.8. Use the result of Exercise 5.8(1) and the method of Example 7.7 to compute all Culler-Shalen seminorms for the (p,q) torus knot exterior.

7.3. **The hyperbolic case.** Suppose that M a hyperbolic knot manifold and let ρ_0 be a lift of the holonomy representation $\rho_M : \pi_1(M) \to PSL(2,\mathbb{C})$ to $SL(2,\mathbb{C})$ (see Example 4.2(1)). Let X_0 be an irreducible component of $X(M)$ containing the character of a discrete faithful representation $\rho_0 : \pi_1(M) \to SL(2,\mathbb{C})$. It is a consequence of Mostow-Prasad rigidity and work of Thurston that

- X_0 is a curve, and
- $\|\cdot\|_{X_0}$ is a norm.

See [CGLS, Proposition 1.1.1]. Moreover, the curve X_0, called a *canonical curve* of M, is essentially unique. Indeed, rigidity shows that ρ_M is unique up to conjugation in $\mathrm{Isom}(\mathbb{H}^3)$ (see Example 3.10(1)) and this implies that the only ambiguity in X_0 comes from the choice of lift of ρ_M and the choice of orientation on M. The finite set of lifts of ρ_M is easily understood and an orientation change of M corresponds to replacing X_0 by a curve consisting of the complex conjugates of the characters in X_0. The norm $\|\cdot\|_{X_0}$ is independent of the choice of canonical curve X_0.

Definition 7.9. A *strict boundary slope* of M is a slope α on ∂M for which there is a connected essential surface F in M which is neither a fibre nor a semi-fibre[2] in M and whose boundary is a non-empty union of curves of slope α.

Proposition 7.10. *Suppose that M is a hyperbolic knot manifold and $X_0 \subset X(M)$ a canonical curve. Then the unit ball B of $\|\cdot\|_{X_0}$ is a finite-sided balanced polygon whose vertices are rational multiples of strict boundary slopes of M.*

Proof. Let $x_1, x_2, \ldots, x_n$ be the ideal points of X_0 whose associated homomorphisms $\phi_{x_j} : H_1(\partial M;\mathbb{R}) \to \mathbb{R}$ are non-zero so that

$$\|\alpha\|_{X_0} = \sum_{j=1}^{n} |\phi_{x_j}(\alpha)|$$

Since $\|\cdot\|_{X_0}$ is a norm, Exercise 7.5(b) shows that B of $\|\cdot\|_{X_0}$ is a finite-sided balanced polygon whose vertices lie on one of the lines $\ker(\phi_j)$. Lemma 7.4(2) shows that $\ker(\phi_j)$ is generated by the class of a boundary slope, so as $\phi_{x_j}(H_1(\partial M)) \subset \mathbb{Z}$, the vertices of B are rational multiples of boundary slopes of M. For the proof that they are strict boundary slopes, see [CGLS, Proposition 1.2.7]. $\square$

Example 7.11. The exterior M of the figure eight knot is a hyperbolic knot manifold whose character variety decomposes as a union of two irreducible curves $X(M) = X_0 \cup X_1$, where $X_0 = X^{irr}(M)$ and $X_1 = X^{red}(M)$ (Example 5.8(2)). As the only curve in $X^{irr}(M)$, X_0 contains the character of the lift of a holonomy representation (Example 4.2(1)) and therefore $\|\cdot\|_{X_0}$ is a norm whose balls are finite-sided balanced polygons with vertices lying on lines in $H_1(\partial M;\mathbb{R})$ determined by certain strict boundary slopes of M (Proposition 7.10). There are only two strict boundary slopes on ∂M and they are represented by the classes $-4\mu+\lambda, 4\mu+\lambda$ ([HT]). Thus the $\|\cdot\|_{X_0}$-norm balls are parallelograms whose vertices lie on these two lines.

There is an orientation reversing homeomorphism $r : M \to M$ which reverses the orientation of the meridional class and determines an algebraic isomorphism $r^* : X_0 \to X_0, \chi_\rho \mapsto \chi_{\rho \circ r_\#}$. Then $f_{\mu^{-n}\lambda^m} = f_{\mu^n\lambda^m} \circ r^*$ so that $\|-n\mu + m\lambda\|_{X_0} = \|n\mu + m\lambda\|_{X_0}$. Hence the $\|\cdot\|_{X_0}$-norm balls are rectangles whose sides are parallel to the μ and λ axes. Figure 2 depicts the norm ball of radius $\|\mu\|_{X_0}$.

To determine $\|\cdot\|_{X_0}$ precisely, recall from Example 5.8(2) the presentation $\pi_1(M) = \langle \gamma_1, \gamma_2 \mid w\gamma_1 = \gamma_2 w\rangle$, where $w = [\gamma_1^{-1}, \gamma_2]$ and the embedding

$$X_0 \to \mathbb{C}^2, \chi_\rho \mapsto (\chi_\rho(\gamma_1), \chi_\rho(\gamma_1\gamma_2^{-1}))$$

whose image is $Y_0 = \{(x,y) \mid y^2 - x^2y + x^2 + y - 1 = 0\} \subset \mathbb{C}^2$. Then for $(x,y) \in Y_0$ we have $x^2 = 1 + \frac{y^2}{y-1}$. On the other hand, γ_1 is a meridional class, so $x = I_\mu$. Hence $f_\mu : X_0 \to \mathbb{C}$ corresponds to the map

[2]A properly embedded surface F in M is a fibre if it has a tubular neighbourhood $N(F) \cong F \times [-1,1]$ for which the closure of its complement is connected and homeomorphic to $F \times [-1,1]$. It is a semi-fibre if it is separating and the closure of each component of $M \setminus N(F)$ is a twisted I-bundle.

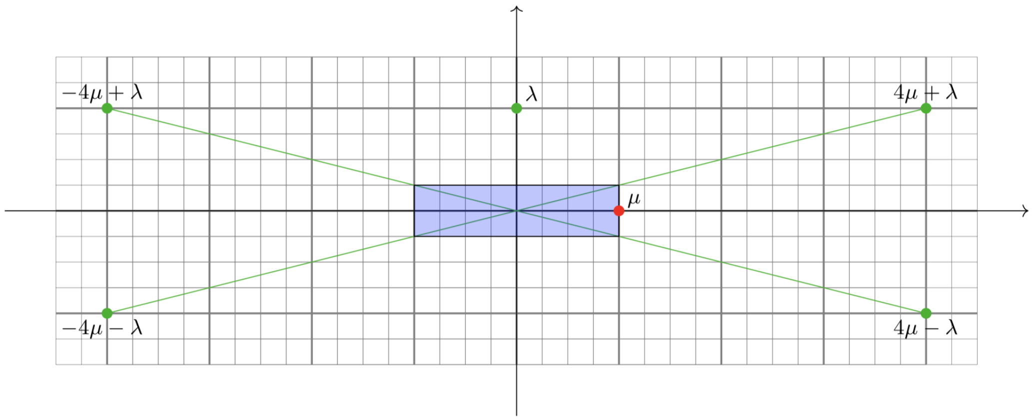

FIGURE 2. The norm ball of the figure eight knot

$Y_0 \to \mathbb{C}, (x,y) \mapsto x^2 - 4 = -3 + \frac{y^2}{y-1}$, which has degree 4. Thus $\|\mu\|_{X_0} = 4$. We leave it to the reader to verify that for $n\mu + m\lambda \in H_1(\partial M)$,

$$\|n\mu + m\lambda\|_{X_0} = 2|n + 4m| + 2|n - 4m|$$

Proposition 7.10 leads to a proof of the hyperbolic case of the *weak Neuwirth conjecture*, which states that the exterior of a non-trivial knot K in the 3-sphere contains a separating essential surface with non-empty boundary. Culler and Shalen proved it in full generality in [CS2].

Theorem 7.12. (Culler-Shalen) *The exterior of a hyperbolic knot $K \subset S^3$ contains an essential separating surface with non-empty boundary.*

Proof. Proposition 7.10 shows that K has at least two boundary slopes and at least one of them, α say, is not the longitudinal slope. Let F be an essential surface properly embedded in M with non-empty boundary of slope α. Orient F and endow the components $C_1, C_2, \ldots, C_m$ of ∂F with the induced orientations. Then for each i there is an $\varepsilon_i \in \{\pm 1\}$ such that $[C_i] = \varepsilon_i \alpha$ and therefore $0 = [\partial F] = (\sum_i \varepsilon_i)\alpha \in H_1(M)$. Since α is not longitudinal, it is non-zero in $H_1(M) \cong \mathbb{Z}$. Hence $\sum_i \varepsilon_i = 0$, so $[\partial F] = 0$ as a class in $H_1(\partial M)$. Then as $H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M)$ is injective, $[F] = 0 \in H_2(M, \partial M)$. It follows that F separates M. $\square$

7.4. **Culler-Shalen seminorms of curves associated to Dehn fillings.** Let $p, q \geq 3$ be coprime integers and M the exterior of the (p,q) torus knot K. Then M admits a Seifert fibre structure with fibre slope $h = pq\mu + \lambda$, where μ and λ are meridional and longitudinal classes for K. It is known that $M(h) \cong L(p,q) \# L(q,p)$ ([Mo]), so $\pi_1(M)$ admits an epimorphism $\varphi : \pi_1(M) \to \mathbb{Z}/p * \mathbb{Z}/q$ and therefore there is an injective morphism of algebraic sets $X(\mathbb{Z}/p * \mathbb{Z}/q) \to X(M), \chi \mapsto \chi \circ \varphi$. It follows from Exercise 5.6(1) that if $p, q \geq 3$, $X^{irr}(\mathbb{Z}/p * \mathbb{Z}/q)$ contains curves of characters. If $X_0 \subset X^{irr}(M)$ is the image of such a curve, it is clear that $\|h\|_{X_0} = 0$, so Exercise 7.6 implies that there is an integer $s \geq 0$ such that $\|\alpha\|_{X_0} = s|\alpha \cdot h|$ for each $\alpha \in H_1(\partial M)$. In fact, arguing as in Example 7.7 shows that $s = 2$.

More generally, if M is a knot manifold and $\alpha \in H_1(\partial M)$ is a primitive class for which $X^{irr}(M(\alpha))$ is positive dimensional, we can find a curve $X_0 \subseteq \text{image}(X^{irr}(M(\alpha)) \to X^{irr}(M))$ such that $\|\alpha\|_{X_0} = 0$

and therefore

$$\|\beta\|_{X_0} = s|\beta \cdot \alpha|$$

for some integer $s \geq 0$. Example 4.2(2) can be used to show that Seifert fibred Dehn fillings often lead to curves X_0 of the sort that we are discussing.

When $s > 0$, $\|\beta\|_{X_0}$ is essentially the absolute value of the algebraic intersection number of α and β, so any constraints on $\|\beta\|_{X_0}$ yield constraints on $|\alpha \cdot \beta|$, a fact that we will exploit in §10.

Even if $s = 0$, significant topological information can be obtained. In this case the ideal points of X_0 will yield closed, essential surfaces (Lemma 7.4), often with special properties. For instance, if M is the exterior of a hyperbolic knot in the 3-sphere and $M(\alpha)$ is a Haken Seifert fibred manifold, there is an $X_0 \subset X^{irr}(M(\alpha)) \subseteq X^{irr}(M)$ and a closed essential surface associated to an ideal point of X_0 which stays essential in $M(\beta)$ whenever $\Delta(\alpha, \beta) > 1$ (see [BZ2, Proposition 4.10] and [BZ2, Claim, page 486]).

8. The cyclic surgery theorem

8.1. **Exceptional fillings of hyperbolic knot manifolds.** An *exceptional Dehn filling slope* of a hyperbolic knot manifold M is a slope α such that $M(\alpha)$ is not hyperbolic. By geometrisation, a filling is exceptional if and only if it is either a reducible manifold, a toroidal manifold, or a Seifert fibred manifold which contains no essential vertical tori[3]. The latter is the family of *small Seifert manifolds* and includes all manifolds with cyclic or finite fundamental groups.

Understanding the set of exceptional fillings slopes

$$\mathcal{E}(M) = \{\alpha \mid M(\alpha) \text{ is not hyperbolic}\}$$

has been of long-standing interest, particularly owing to Thurston's hyperbolic Dehn surgery theorem which shows that it is a finite set. A basic questions is:

Question 8.1. How large can $\mathcal{E}(M)$ be and how are its elements distributed in the surgery plane $H_1(\partial M; \mathbb{R})$?

Results on this question are often phrased in terms of the *distance* between slopes α and β which, when they are represented by primitive elements of $H_1(\partial M)$, is given by

$$\Delta(\alpha, \beta) = |\alpha \cdot \beta|$$

(As above, $|\alpha \cdot \beta|$ denotes the absolute value of the algebraic intersection number between α and β.) The reader will verify that $\Delta(\alpha, \beta) = 0$ if and only if α and β represent the same slope and $\Delta(\alpha, \beta) = 1$ if and only if α and β span $H_1(\partial M)$.

An a priori bound on the distance between elements of a set of slopes $\mathcal{S}$ leads to bounds on the size of $\mathcal{S}$. Suppose, for instance, that the distance between the elements of $\mathcal{S}$ is bounded above by $n \geq 1$. Since no primitive element of $H_1(\partial M)$ lies in the kernel of the quotient homomorphism $\kappa : H_1(\partial M) \to H_1(\partial M)/(n+1)H_1(\partial M) = H_1(\partial M; \mathbb{Z}/(n+1))$, κ induces an injective function $\bar{\kappa} : \mathcal{S} \to H_1(\partial M; \mathbb{Z}/(n+1))^*/\{\pm 1\}$. Moreover, the assumed distance bound implies that $\bar{\kappa}$ is injective, so

$$\#\mathcal{S} \leq \begin{cases} 3 & \text{if } n = 1 \\ n(n+2)/2 & \text{if } n \geq 2 \end{cases}$$

[3]A vertical torus in a Seifert fibre space is a torus that is the union of regular Seifert fibres.

An a priori bound on the distance between the elements of $\mathcal{S}$ also constrains its relative positioning in the surgery plane. For instance, a bound of 1 implies that $\mathcal{S}$ is contained in a set of the form $\{\alpha, \beta, \alpha+\beta\}$, where α, β forms a basis of $H_1(\partial M)$.

Set

$$\Delta(M) = \max\{\Delta(\alpha, \beta) \mid \alpha, \beta \in \mathcal{E}(M)\}$$

Here is another basic question:

Question 8.2. What is the topology of M when either $\#\mathcal{E}(M)$ or $\Delta(M)$ is relatively large?

Conjecture 8.3 (C. McA. Gordon)**.** *For any hyperbolic knot manifold M, $\#\mathcal{E}(M) \leq 10$ and $\Delta(M) \leq 8$. Moreover, excluding four explicit possibilities for M, $\#\mathcal{E}(M) \leq 7$ and $\Delta(M) \leq 5$.*

When M is the exterior of the figure eight knot, $\#\mathcal{E}(M) \leq 10$ and $\Delta(M) \leq 8$. See [Go2, Conjecture 3.4] for a description of the other three hyperbolic knot manifolds M with $\Delta(M) > 5$.

Much progress has been made towards verifying Conjecture 8.3. For instance, Lackenby and Meyerhoff have shown that $\#\mathcal{E}(M) \leq 10$ and $\Delta(M) \leq 8$ for any hyperbolic knot manifold M ([LM]), while the author with Gordon and Zhang have verified it if the first Betti number of M is at least 2 [BGZ1]. We will discuss this further in §11.

8.2. **The cyclic surgery theorem.** A slope on the boundary of knot manifold is called a *cyclic surgery slope* if the associated Dehn filling has cyclic fundamental group.

The main result of [CGLS] is:

Theorem 8.4. (The cyclic surgery theorem) *Let M be a non-Seifert fibred knot manifold and α, β be two cyclic surgery slopes on ∂M. Then $\Delta(\alpha, \beta) \leq 1$. Hence there are at most three Dehn fillings of M with cyclic fundamental groups.*

The exclusion of Seifert fibred knot manifolds is necessary; torus knot exteriors admit infinitely many cyclic surgery slopes and the set of distances between these slopes is unbounded.

Before discussing the proof of the cyclic surgery theorem, we describe two immediate consequences.

Let $K \subset S^3$ be a non-trivial knot and M its exterior. The *meridional slope* of K is the unique slope μ on ∂M which is homologically trivial in the solid torus $\overline{S^3 \setminus M}$. Since $M(\mu) \cong S^3$, μ is a cyclic surgery slope.

The *longitudinal slope* of K is the unique slope λ on ∂M which is homologically trivial in $H_1(M)$. In other words, there is a compact, connected, orientable surface S properly embedded in M whose boundary is a simple closed curve of slope λ.

An orientation of the knot K determines orientations for the meridional and longitudinal slopes of K using the right-hand rule. Reversing the orientation of K simultaneously reverses those of μ and λ.

The Mayer-Vietoris sequence can be used to show that μ and λ form a basis of $H_1(\partial M)$ and as such, $\Delta(\mu, \lambda) = 1$. Each slope α on ∂M corresponds to a pair of primitive classes $\pm(p\mu + q\lambda)$ in $H_1(\partial M)$, where p and q are coprime integers. In this way we can identify the set of slopes on ∂M with $\mathbb{Q} \cup \{\frac{1}{0}\}, \pm(p\mu + q\lambda) \longleftrightarrow \frac{p}{q}$.

An *integer slope* α on ∂M is one represented by $p\mu + \lambda \longleftrightarrow p \in \mathbb{Z}$. Equivalently, $\Delta(\alpha, \mu) = 1$.

Corollary 8.5. *Let K be a non-torus knot in S^3. If $\alpha \neq \mu$ is a cyclic surgery slope, it is an integer slope. Further, there are at most two such slopes, and if two, they are successive integers.* □

The corollary is sharp; if K is the $(-2,3,7)$-pretzel knot, then 18-surgery and 19-surgery yield the lens spaces $L(18,5)$ and $L(19,7)$.

The Property P Conjecture contends that no non-trivial surgery on a non-trivial knot in the 3-sphere yields a simply-connected manifold. Since the first homology of a manifold obtained by p/q-surgery on a knot $K \subset S^3$ is cyclic of order p, if such a surgery yields a simply-connected manifold, then $p = \pm 1$. Moser [Mo] verified the Property P Conjecture for torus knots, and this combines with Theorem 8.4 to yield,

Corollary 8.6. *Let K be a non-trivial knot in S^3. Then there is at most one non-trivial slope α for which $M_K(\alpha)$ is simply connected and any such slope is contained in $\{\pm 1\}$.* □

The Property P conjecture was later verified in full generality by Kronheimer and Mrowka using gauge theoretic methods. See [KM].

8.3. **The proof of the cyclic surgery theorem.** The proof splits into four cases:

(1) M admits an essential torus;
(2) $b_1(M) \geq 2$;
(3) $b_1(M) = 1$ and at least one of α, β is a strict boundary slope;
(4) M is hyperbolic and neither α nor β is a strict boundary slope.

The intersection graph methods of Gordon-Luecke are key to the proofs of the first three cases (see [CGLS, Theorems 2.0.1, 2.0.2, 2.0.3]). For instance, in the second case the first Betti numbers of $M(\alpha)$ and $M(\beta)$ are at least 1, so they admit closed, orientable, non-separating, incompressible surfaces which are necessarily 2-spheres, as their fundamental groups are cyclic. Thus $M(\alpha)$ and $M(\beta)$ are reducible and the bound $\Delta(\alpha, \beta) \leq 1$ follows from Gordon's lectures in this volume. Here we focus on the last case.

Assume that M is hyperbolic and α is a cyclic filling slope, but not a strict boundary slope. Let $\|\cdot\|_{X_0}$ be the Culler-Shalen norm associated to a canonical curve X_0 in $X^{irr}(M)$ (see §7.3). For $r > 0$, set

$$B_r = \{v \in H_1(\partial M; \mathbb{R}) \mid \|v\|_{X_0} \leq r\}$$

The key to proving our case of the cyclic surgery theorem is found in the following proposition.

Proposition 8.7. *Suppose that X_0 a canonical curve of a hyperbolic knot manifold M and set $s = \min\{\|\delta\|_{X_0} \mid \delta \in H_1(\partial M) \setminus \{0\}\}$. If α is a cyclic filling slope of M, though not a strict boundary slope, then $\|\alpha\|_{X_0} = s$.*

Examples 8.8.

(1) The meridional slope μ of any knot K in the 3-sphere is a cyclic surgery slope and in the case that K is the figure eight knot with exterior M, μ is not a strict boundary slope. We saw in Example 7.11 that $\|\mu\|_{X_0}$ realises the minimal non-zero value 4 of the restriction of $\|\cdot\|_{X_0}$ to $H_1(\partial M)$, where X_0 is a canonical curve of the hyperbolic structure on M.

(2) The exterior M of the $(-2,3,7)$ pretzel knot has three strict boundary slopes $16\mu+\lambda, 37\mu+2\lambda$, and $20\mu+\lambda$ (depicted by the green dots in Figure 3). As noted above, it admits three cyclic filling slopes

$\mu, 18\mu+\lambda$, and $19\mu+\lambda$ (the red dots), each of which must realise the minimal value s of $\|\cdot\|_{X_0}$ on $H_1(\partial M)\setminus\{0\}$, where X_0 is a canonical curve of the hyperbolic structure on M. It is shown in [BMZ] that $s=12$ and the $\|\cdot\|_{X_0}$-ball of radius 12 is as shown in Figure 3.

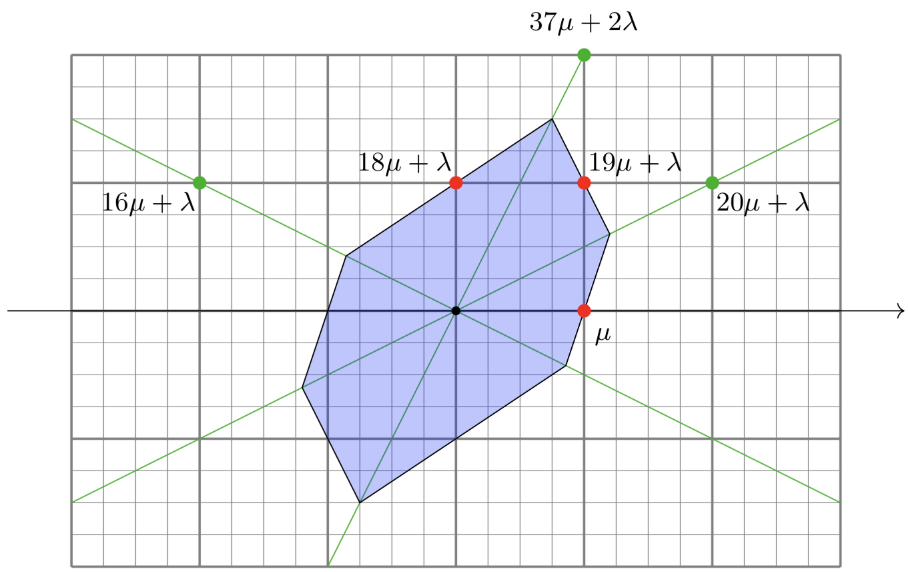

FIGURE 3. The norm ball of the figure $(-2,3,7)$-pretzel knot with the cyclic surgery slopes depicted by the solid red dots.

Assuming the truth of Proposition 8.7, we can complete the proof of the case of the cyclic surgery theorem that we are considering through the use of Minkowski's theorem on convex bodies [Ca].

The proposition implies that if α and β are cyclic filling slopes, but not strict boundary slopes, then $\alpha,\beta\in\partial B_s$. By construction, $\text{int}(B_s)\cap H_1(\partial M)=\{0\}$, so if we identify the pair $(H_1(\partial M,\mathbb{R}),H_1(\partial M))$ with $(\mathbb{R}^2,\mathbb{Z}^2)$, Minkowski's theorem implies that $\text{area}(B_s)\le 4$. If $P(\alpha,\beta)$ denotes the parallelogram in $H_1(\partial M,\mathbb{R})$ with vertices $\pm\alpha,\pm\beta$, it is an exercise in plane geometry to show that $\text{area}(P(\alpha,\beta))=2\Delta(\alpha,\beta)$. On the other hand, $P(\alpha,\beta)\subseteq B_s$ so that

$$\Delta(\alpha,\beta)=\frac{1}{2}\text{area}(P(\alpha,\beta))\le\frac{1}{2}\text{area}(B_s)\le\frac{1}{2}(4)=2$$

with $\Delta(\alpha,\beta)=2$ implying that $B_s=P(\alpha,\beta)$. In the latter case, α is a vertex of B_s and so is a strict boundary slope (Proposition 7.10), contrary to our assumptions. Thus $\Delta(\alpha,\beta)\le 1$.

Proof of Proposition 8.7. Recall that $\|\alpha\|_{X_0}=\sum_{x\in\widetilde{X}_0}\Pi_x(f_\alpha)=\text{degree}(f_\alpha:\widetilde{X}_0\to\mathbb{C}P^1)$. We can equally well calculate the degree of f_α by counting zeros:

$$\|\alpha\|_{X_0}=\sum_{x\in\widetilde{X}_0}Z_x(f_\alpha),$$

where $Z_x(f_\alpha)$ denotes the multiplicity of $x\in\widetilde{X}_0$ as a zero of f_α. Hence the minimality of $\|\alpha\|_{X_0}$ will follow from the claim that

$$Z_x(f_\alpha)\le Z_x(f_\delta)\text{ for all }\delta\in H_1(\partial M)\setminus\{0\}\text{ and }x\in\widetilde{X}_0$$

Suppose otherwise that $Z_x(f_\alpha) > Z_x(f_\delta)$ for some $\delta \in H_1(\partial M) \setminus \{0\}$ and $x \in \widetilde{X}_0$. Then as $Z_x(f_\alpha) > Z_x(f_\delta) \geq 0$,

$$f_\alpha(x) = 0$$

We consider two cases.

Case 1. x is an ideal point of $\widetilde{X}_0$.

By Lemma 7.1, there is a homomorphism $\phi_x : H_1(\partial M) \to \mathbb{Z}$ for which $|\phi_x(\gamma)| = \Pi_x(f_\gamma)$ for $\gamma \in \pi_1(\partial M)$. Then $\phi_x(\alpha) = 0$, since $f_\alpha(x) = 0$. If $\phi_x \not\equiv 0$, α is a vertex of B_s and hence a strict boundary slope by Proposition 7.10, contrary to our assumptions. Thus $\phi_x \equiv 0$ and so there is a closed essential surface S in M which is dual to the action of $\pi_1(M)$ on $T(\omega_x)$ (Lemma 7.4(1)). Since M is hyperbolic, the genus of S is at least 2. This already leads to a contradiction when M contains no closed essential surfaces (i.e., M is *small*). In the general case, the condition $Z_x(f_\alpha) > Z_x(f_\delta)$ is used to show that $\pi_1(\partial M)$ acts trivially on the link of some vertex of $T(\omega_x)$ ([CGLS, Lemma 1.6.5]). A delicate argument then shows that if S is chosen to minimize a standard complexity function, it stays essential in $M(\alpha)$ ([CGLS, Proposition 1.6.1]), which is impossible since $\pi_1(M(\alpha))$ is cyclic.

Case 2. x is an ordinary point of $\widetilde{X}_0$.

Recall that R_0 is the unique conjugation invariant 4-dimensional subvariety of $R(M)$ such that $t(R_0) = X_0$. Set $t_0 = t|_{R_0}$ and note that since R_0 is a 4-dimensional affine algebraic set and t_0 is regular and surjective, dimension$(t_0^{-1}(\chi)) = 3$ for each $\chi \in X_0$ ([Mu, Theorem 3.1.3]).

Let $\rho \in R_0$ be an arbitrary element of $t_0^{-1}(\nu(x))$. We have $\mathrm{tr}(\rho(\alpha)) = \chi_\rho(\alpha) = \pm 2$, since $f_\alpha(x) = 0$, and therefore $\rho(\alpha)$ is either parabolic or $\pm I$. Our hypothesis that $Z_x(f_\alpha) > Z_x(f_\delta)$ combines with the fact that α and δ commute in $\pi_1(M)$ to show that $\rho(\alpha) = \pm I$ ([CGLS, Proposition 1.5.2]). Hence ρ determines a homomorphism

$$\bar{\rho} : \pi_1(M(\alpha)) \to PSL(2, \mathbb{C})$$

which, by hypothesis, has cyclic image. It follows that $\bar{\rho}$ can be conjugated into the subgroup of diagonal matrices $\{\pm \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^*\}$ of $PSL(2, \mathbb{C})$ or the subgroup of upper triangular parabolic matrices $\{\pm \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C}\}$. A similar conclusion then holds for the image of ρ, and therefore the centraliser in $SL(2, \mathbb{C})$ of image(ρ) has dimension at least 1. It follows that the orbit $\mathcal{O}(\rho)$ of ρ under conjugation has dimension 2 or less, and so is nowhere dense in $t_0^{-1}(\nu(x))$. We show that this leads to a contradiction.

Suppose first that $\nu(x)(\pi_1(M)) \not\subset \{\pm 2\}$. Then from the previous paragraph we see that the image of any $\rho' \in t_0^{-1}(\nu(x))$ conjugates into the subgroup of diagonal matrices of $SL(2, \mathbb{C})$. Hence, the kernel of ρ' is given by $K = \chi_{\rho'}(x)^{-1}(2) = \nu(x)^{-1}(2)$, which is independent of ρ'.

Fix $\rho \in t_0^{-1}(\nu(x))$ and choose $\gamma_0 \in \pi_1(M)$ such that $\bar{\rho}(\gamma_0)$ generates the image of $\bar{\rho}$. Then either the image of ρ is generated by $\rho(\gamma_0)$ or or it is generated by $\rho(\gamma_0)$ and $-I$. Hence $H := \rho^{-1}(\langle \rho(\gamma_0) \rangle)$ has index 1 or 2 in $\pi_1(M)$, and if 2, $\pi_1(M)$ is generated by H and any $\gamma_1 \in \pi_1(M)$ satisfying $\rho(\gamma_1) = -I$.

We claim that $t_0^{-1}(\nu(x))$ coincides with the orbit $\mathcal{O}(\rho)$ of ρ. If so, we arrive at a contradiction since $\mathcal{O}(\rho)$ is nowhere dense in $t_0^{-1}(\nu(x))$.

To see that $t_0^{-1}(\nu(x)) = \mathcal{O}(\rho)$, suppose that $\rho' \in t_0^{-1}(\nu(x))$ is arbitrarily chosen. After conjugating, we can assume that both ρ and ρ' have images contained in the group of diagonal matrices. Further, since $\chi_{\rho'}(\gamma_0) = \chi_\rho(\gamma_0)$, up to performing a further conjugation of ρ' we can suppose that $\rho'(\gamma_0) = \rho(\gamma_0)$.

For each $\gamma \in H$ there is an integer m and $\kappa \in K$ such that $\gamma = \gamma_0^m \kappa$. Then

$$\rho'(\gamma) = \rho'(\gamma_0^m \kappa) = \rho'(\gamma_0)^m = \rho(\gamma_0)^m = \rho(\gamma_0^m \kappa) = \rho(\gamma)$$

Thus $\rho'|_H = \rho|_H$ and so if $H = \pi_1(M)$, $t_0^{-1}(\nu(x))$ is indeed $\mathcal{O}(\rho)$. Otherwise, H has index 2 in $\pi_1(M)$ and if $\gamma_1 \in \pi_1(M)$ satisfies $\rho(\gamma_1) = -I$, then $\pi_1(M)$ is generated by H and γ_1. Further, as $\chi_{\rho'}(\gamma_1) = \chi_\rho(\gamma_1) = -2$, we have $\rho'(\gamma_1) = -I$. Then once again, $\rho' = \rho$ and therefore $t_0^{-1}(\nu(x)) = \mathcal{O}(\rho)$, which is what we needed to prove in the case that $\nu(x)(\pi_1(M)) \not\subset \{\pm 2\}$.

Suppose next that the image of $\nu(x)$ is contained in $\{\pm 2\}$. Here, $t_0^{-1}(\nu(x))$ may not be contained in a single orbit, but we claim that it is contained in a countable union of orbits, which contradicts the Baire category theorem since we noted above that each orbit is nowhere dense in $t_0^{-1}(\nu(x))$.

To prove the claim, fix $\rho \in t_0^{-1}(\nu(x))$ and $\gamma_0 \in \pi_1(M)$ such that $\bar{\rho}(\gamma_0)$ generates the image of $\bar{\rho}$. After conjugating, we can assume that $\bar{\rho}(\gamma_0) \in PSL(2, \mathbb{C})$ is either $\pm I$ or $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It follows that up to conjugation, the image of ρ is contained in the group $\{\begin{pmatrix} \varepsilon & k \\ 0 & \varepsilon \end{pmatrix} \mid \varepsilon \in \{\pm 1\}, k \in \mathbb{Z}\}$. If $\gamma_1, \gamma_2, \ldots, \gamma_n$ is a generating set for $\pi_1(M)$ and $\varepsilon_i = \nu(x)(\gamma_i)/2 \in \{\pm 1\}$, then for each $1 \le i \le n$,

$$\rho(\gamma_i) = \begin{pmatrix} \varepsilon_i & k_i \\ 0 & \varepsilon_i \end{pmatrix},$$

for some $k_i \in \mathbb{Z}$. Since there are only countably many choices for such k_i, $t_0^{-1}(\nu(x))$ is the union of countable many orbits, which completes the proof. □

The next two sections describe further developments and applications of character variety methods to surgery theory.

9. The finite surgery theorem

Cameron McA. Gordon conjectured a companion result to the cyclic surgery theorem ([Go1]), which was later verified by Xingru Zhang and the author ([BZ3]).

Theorem 9.1. (The finite surgery theorem) *If M is a hyperbolic knot manifold, there are at most five slopes α on ∂M with $\pi_1(M(\alpha))$ finite. Further, the distance between any two such slopes is at most* 3.

The theorem is sharp, as the following example shows.

Example 9.2. The *figure eight sister knot manifold* M is obtained by $+5$-filling of a boundary component of the right-handed Whitehead link. Its Culler-Shalen norm ball B_s of radius $s = \min\{\|\delta\|_{X_0} \mid \delta \in H_1(\partial M) \setminus \{0\}\}$ was calculated in [BZ1, Example 10.5] and is depicted in Figure 4 as the heavier shaded parallelogram.

The $\mu, \mu+\lambda$ and $2\mu+\lambda$ (the red dots in Figure 4) Dehn fillings of M are lens spaces and the $3\mu+\lambda, 3\mu+2\lambda$ (the blue dots) Dehn fillings yield Seifert manifolds with base orbifolds $S^2(2, 3, 3)$, hence have finite fundamental groups (see §9.1 below). The reader will verify that the distance between any two of these

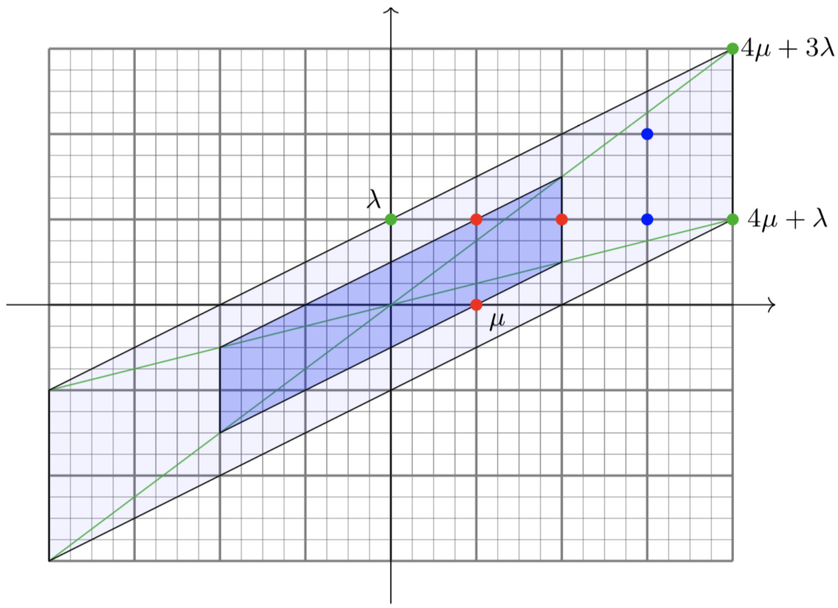

FIGURE 4. The Culler-Shalen norm ball of the figure eight sister knot with the cyclic surgery slopes depicted by the small solid red dots

five slopes is at most 3 and that $3\mu + \lambda, 3\mu + 2\lambda$ are the only two of distance 3. The set of exceptional slopes, $\mathcal{E}(M)$, is the union of these five finite filling slopes and the slopes $\lambda, 4\mu + \lambda, 4\mu + 3\lambda$ (the green dots), which yield toroidal Dehn fillings. It is interesting to note that $\mathcal{E}(M)$ is contained in $2B_s$.

There is a companion result for toroidal knot manifolds [BZ1, Theorem 1.2]: *If M is a toroidal knot manifold, though not the union of a cable space and a twisted I-bundle over the Klein bottle, then there are at most six slopes α on ∂M with $\pi_1(M(\alpha))$ finite and the distance between any two such slopes is at most* 5. This result is sharp; see the discussion following [BZ1, Theorem 1.2].

The proof of Theorem 9.1 necessitates the introduction of a number of new ideas, which we describe in the following two subsections. For now, note that we can suppose that M has first Betti number 1; otherwise it has no finite filling slopes. Further, intersection graph methods can be used to reduce the analysis to the case of finite filling slopes which are not strict boundary slopes (see [CGLS, Theorem 2.0.3]). In this case, if $\|\cdot\|_{X_0}$ denotes the Culler-Shalen norm given by a canonical curve X_0 in $X^{irr}(M)$, any cyclic filling class would realise the minimal non-zero value of the restriction of $\|\cdot\|_{X_0}$ to $H_1(\partial M)$ (Proposition 8.7). A key component of the proof of the finite surgery theorem will be to show that the though the norms of finite filling classes are not necessarily minimal, they are strongly constrained.

9.1. **Finite filling classes and bounds on the values that the Culler-Shalen norm takes on them.** A closed, connected 3-manifold has a finite fundamental group if and only if it is a lens space or admits a Seifert structure with base orbifold $\mathcal{B} = S^2(a, b, c)$, necessarily unique, where $a, b, c \geq 2$ is a Platonic triple. Moreover,

$$\pi_1(\mathcal{B}) \cong \begin{cases} \text{the dihedral group } D_n \text{ of order } 2n \geq 4 & \text{if } \mathcal{B} = S^2(2,2,n) \\ \text{the tetrahedral group } T \text{ of order } 12 & \text{if } \mathcal{B} = S^2(2,3,3) \\ \text{the octahedral group } O \text{ of order } 24 & \text{if } \mathcal{B} = S^2(2,3,4) \\ \text{the icosahedral group } I \text{ of order } 60 & \text{if } \mathcal{B} = S^2(2,3,5) \end{cases}$$

We say that a finite filling slope α is of

- C-type (for cyclic) if $M(\alpha)$ is a lens space;
- D-type (for dihedral) if $\mathcal{B} = S^2(2,2,n)$;
- T-type (for tetrahedral) if $\mathcal{B} = S^2(2,3,3)$;
- O-type (for octahedral) if $\mathcal{B} = S^2(2,3,4)$;
- I-type (for icosahedral) if $\mathcal{B} = S^2(2,3,5)$;

It is shown in [BZ3] that if X_0 is a canonical curve of a hyperbolic knot manifold M and α is a finite filling slope on ∂M that is not a strict boundary slope, then

(1) $\|\alpha\|_{X_0}$ realises the minimal non-zero value of $\|\cdot\|_{X_0}$ on a sublattice of $H_1(\partial M)$ containing α whose index, which depends on the type of α, is at most 5;

(2) if s denotes the minimal non-zero value of $\|\cdot\|_{X_0}$, then $\|\alpha\|_{X_0}$ is bounded above by $s, 2s, s+4, s+8$, or $s+12$ when $M(\alpha)$ is, respectively, C-type, D-type, T-type, I-type, or O-type.

The first of these statements is established by applying the ideas of the proof of Proposition 8.7 to the fact that $M(\alpha)$ is finitely covered by a manifold with cyclic fundamental group. Heuristically, the key to the second statement is that $\pi_1(M(\alpha))$ has very few irreducible $SL(2,\mathbb{C})$-characters if α is T-type, O-type, or I-type, though converting this into bounds on the values of $\|\alpha\|_{X_0}$ requires proving that if $Z_x(f_\alpha) > Z_x(f_\delta)$ for some $\delta \in H_1(\partial M) \setminus \{0\}$ and $x \in \widetilde{X}_0$ (cf. the proof of Proposition 8.7), then x is an ordinary point of $\widetilde{X}_0$ (§6.1), $\nu(x)$ is a smooth point of X_0, and $Z_x(f_\alpha) = 2, Z_x(f_\gamma) = 0$. See [BZ1].

It follows that the finite filling slopes which are not strict boundary slopes lie relatively close to the Culler-Shalen norm ball of radius s, B_s. For instance, in Example 9.2 these slopes lie in $2B_s$, where B_s is the darker shaded parallelogram in the accompanying figure. It is proven in [BZ1] that these constraints combine with the geometry of the norm balls of $\|\cdot\|_{X_0}$ to show that there are at most six finite filling slopes and the distance between any two is at most 5. In order to obtain the improved bounds needed for the finite surgery theorem, a reinterpretation of Culler-Shalen norms which carries more subtle information is needed. This is described in the next subsection.

9.2. **The A-polynomial.** Let $i : \partial M \to M$ be the inclusion and $i_\# : \pi_1(\partial M) \to \pi_1(M)$ the induced homomorphism. The map $i^\# : X_0 \to X(\partial M), \chi_\rho \mapsto \chi_{\rho \circ i_\#}$, is a morphism of algebraic sets whose image has closure an irreducible algebraic set $Y_0 \subset X(\partial M)$.

For each $\alpha \in H_1(\partial M) = \pi_1(\partial M)$, set $g_\alpha : Y_0 \to \mathbb{C}, \chi \mapsto \chi(\alpha)^2 - 4$, and note that the following diagram commutes:

$$\begin{tikzcd} X_0 \arrow[r, "i^\#"] \arrow[dr, "f_\alpha"'] & Y_0 \arrow[d, "g_\alpha"] \\ & \mathbb{C} \end{tikzcd}$$

If $\alpha \neq 0$, the fact that $\|\cdot\|_{X_0}$ is a norm implies that g_α is non-constant and therefore Y_0 is a curve. Moreover, $\|\alpha\|_{X_0} = \text{degree}(f_\alpha) = \text{degree}(i^\#)\text{degree}(g_\alpha)$, so the Culler-Shalen norm of X_0 is determined up to a multiplicative factor by the behaviour of the functions g_α on Y_0. To obtain a clearer picture of Y_0, we introduce a construction due to Cooper, Culler, Gillet, Long, and Shalen ([CCGLS]).

Exercise 9.3. Show that each representation $\rho \in R(\partial M)$ is conjugate to a representation with image in the group of upper triangular matrices and each $\chi \in X(\partial M)$ is the character of a representation with diagonal image.

Thus, if $D(\partial M)$ is the algebraic subset $\{\rho \in R(\partial M) \mid \text{image}(\rho) \text{ is diagonal}\}$ of $R(\partial M)$, the map $t_0 : D(\partial M) \to X(\partial M), \rho \mapsto \chi_\rho$, is surjective.

Exercise 9.4. Show that $D(\partial M)$ is an algebraic subset of $R(\partial M)$ and that $t_0 : D(\partial M) \to X(\partial M), \rho \mapsto \chi_\rho$ is a morphism of algebraic sets.

Fix a basis $\{\alpha, \beta\}$ of $\pi_1(\partial M)$. The map $\xi : D(\partial M) \to \mathbb{C}^2$ which sends a representation ρ to the $(1,1)$-entries $(\xi_\alpha(\rho), \xi_\beta(\rho))$ of $\rho(\alpha), \rho(\beta)$, is injective and polynomial with image $\mathbb{C}^* \times \mathbb{C}^*$. Let W_0 denote the curve $t_0^{-1}(Y_0) \subset \mathcal{D}(\partial M)$ and D_0 the algebraic closure of $\xi(W_0)$ in $\mathbb{C} \times \mathbb{C}$. The following diagram summarizes the situation.

$$\begin{tikzcd} & \mathcal{D}(\partial M) \supset W_0 = t_0^{-1}(Y_0) \arrow[r, "\xi|_{W_0}"] \arrow[d, "t_0"] & \xi(W_0) \subset \overline{\xi(W_0)} = D_0 \subset \mathbb{C}^2 \\ X_0 \arrow[r, "i^{\#}"] & Y_0 = \overline{i^{\#}(X_0)} \subset X(\partial M) & \end{tikzcd}$$

Exercise 9.5. Show that $D_0 \cap (\mathbb{C}^* \times \mathbb{C}^*)$ is invariant under involution of which sends (u,v) to (u^{-1}, v^{-1}).

As a plane curve, D_0 is the zero set of a 2-variable complex polynomial $A_{X_0}(u,v)$ with no repeated factors. This polynomial, called the *A-polynomial of* X_0 ([CCGLS]), is uniquely determined up to multiplication by a non-zero complex constant. Exercise 9.5 implies that there are integers $r, s \geq 0$ such that

$$A_{X_0}(u^{-1}, v^{-1}) = \pm u^r v^s A_{X_0}(u,v) \tag{9.2.1}$$

The reader will notice that the only constraint on X_0 needed for the construction of A_{X_0} is that $i^{\#}(X_0)$ be 1-dimensional and modulo this proviso, the construction extends to any curve in the character variety of any knot manifold.

Examples 9.6. Suppose that M is the exterior of a knot K. Orient K and its meridional and longitudinal slopes μ and λ so that λ is parallel to K and μ has linking number $+1$ with K.

(1) The set of reducible characters of $\pi_1(M)$ forms a curve $X_0 \subset X(M)$ parametrised by the representations factoring through $H_1(M)$ and sending μ to $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{C}^*$. It's easy to see then that $Y_0 = \overline{i^{\#}(X_0)}$ is a curve and $\chi(\lambda) = 2$ for each $\chi \in Y_0$. It follows that $\rho(\lambda) = I$ for each $\rho \in W_0 = t_0^{-1}(Y_0)$ and therefore $D_0 = \overline{\xi(W_0)} = \mathbb{C} \times \{1\}$. Hence $A_{X_0}(u,v) = v - 1$.

(2) Suppose that M is the exterior of the right-handed trefoil knot K and X_0 is the curve $X^{irr}(M)$ (see Example 5.7). The slope h of the fibre class of the Seifert fibre structure on M is represented by $h = 6\mu + \lambda$. We saw in Example 7.7 that $\| \cdot \|_{X_0}$ is a non-trivial seminorm, so $Y_0 = \overline{i^{\#}(X_0)}$ is a curve. We also noted that thinking of h as an element of $\pi_1(M)$, we have $\rho(h) = -I$ for each irreducible ρ with $\chi_\rho \in X_0$. Then if $\rho \in W_0 = \overline{t_0^{-1}(Y_0)}$, $\chi_\rho(h) = -2$, so as $\rho(h)$ is a diagonal matrix, $\rho(h) = -I$. Thinking

of h as the element $\mu^6\lambda \in \pi_1(\partial M)$, it follows that $u^6 v = -1$ where u and v are the $(1,1)$-entries of $\rho(\mu)$ and $\rho(\lambda)$. Then $\xi(W_0)$ is the curve $\{(u,v) \in \mathbb{C}^* \times \mathbb{C}^* \mid u^6 v + 1 = 0\}$, so $A_{X_0}(u,v) = u^6 v + 1$.

The condition that $i^{\#}(X_0)$ be 1-dimensional holds quite generally. For instance, given that knot manifolds have only finitely many boundary slopes ([Ha1]), the following exercise shows that if M contains no closed essential surfaces and X_0 is an irreducible algebraic subset of $X(M)$, then $i^{\#}(X_0)$ has dimension 0 or 1.

Exercise 9.7. Suppose that X_0 is an irreducible algebraic subset of $X(M)$ such that $i^{\#}(X_0)$ is 2-dimensional. Let $\theta = m\alpha + n\beta$ be a primitive class in $H_1(\partial M)$ and let D_θ be a curve component of $\{(u,v) \in \mathbb{C}^* \times \mathbb{C}^* \mid u^m v^n = 1\}$. Show that $\overline{(i^{\#})^{-1}(t_0(\xi^{-1}(D_\theta)))} \subset X(M)$ contains an irreducible curve X_1 on which f_θ is constantly zero. Deduce from Lemma 7.4 that either M contains a closed essential surface or θ is a boundary slope.

Let $X_1, X_2, \ldots, X_k$ be the irreducible components of $X(M)$ for which $i^{\#}(X_0)$ has dimension 1 and define the A-polynomial of M to be

$$A_M(u,v) = \prod_{i=1}^{k} A_{X_i}(u,v)$$

It is shown in [CCGLS] that up to multiplication by a non-zero constant, $A_M(u,v)$ has integer coefficients. We can also suppose that their greatest common denominator is 1, so A_M is well-defined up to sign.

Exercise 9.8. Calculate the A-polynomial of the exterior of a torus knot. (See Exercises 5.8 , 7.8 and Example 9.6(2).)

An associated object of particular interest is the *Newton polygon* of $A_M(u,v)$ which, if we write $A_M(u,v) = \sum_{m,n} a_{m,n} u^m v^n$, is defined to be

$$N_M = \text{Convex hull}(\{(m,n) \mid a_{m,n} \neq 0\}) \subset \mathbb{R}^2$$

Equation (9.2.1) implies that N_M is invariant under reflection in some point of $\mathbb{R}^2$. In other words, it is a *balanced* polygon. Moreover, the coefficients of A_M satisfy the following striking property:

- if E is an edge of N_{X_0}, then the *edge polynomial* $h_E(z) = \sum_{(m,n)\in E} a_{m,n} z^n$ is a product of cyclotomic polynomials ([CCGLS, Proposition 5.10], [CL]);

As constructed, A_M depends on the choice of basis $\{\alpha, \beta\}$ for $H_1(\partial M)$, though in the case that M is the exterior of a knot in an oriented integer homology sphere (e.g. S^3), the canonical choice is to take α to be a meridional class and β a longitudinal class (coherently oriented). In general, there is no canonical choice of basis, but the reader will verify that up to sign,

$$\sum_{m,n} a_{m,n}[n\alpha + m\beta] \in \mathbb{Z}[H_1(\partial M)]$$

is independent of basis, so provides an essentially well-defined expression for A_M as an element $\sum_\theta a_\theta[\theta] \in \mathbb{Z}[H_1(\partial M)]$. With this interpretation, N_M becomes

$$N_M = \text{Convex hull}(\{\theta \mid a_\theta \neq 0\}) \subset H_1(\partial M; \mathbb{R})$$

and [CCGLS, Theorem 3.4] is

- the edges of N_M are parallel to boundary slopes of M.

The following related theorem is an essential ingredient in the proof of the finite surgery theorem.

Theorem 9.9. ([BZ3, §8]) *Let X_0 be an irreducible curve in $X(M)$ for which $\| \cdot \|_{X_0}$ is a norm. Then the norm balls B_r of $\| \cdot \|_{X_0}$ are dual to the Newton polygon N_{X_0} of A_{X_0}. That is each edge of N_{X_0} is parallel to a line running though opposite vertices of B_r and vice versa.*

Starting from a presentation for $\pi_1(M)$, the A-polynomial of a knot manifold M can be computed, in principle, using classical elimination theory ([CCGLS, §7]), though this has proven challenging even for machine calculation. The appendix of [CCGLS] lists the A-polynomials of some twenty knots with low crossings numbers.

Examples 9.10.

(1) The $SL(2, \mathbb{C})$-character variety of the figure 8 knot exterior M is the union of two irreducible curves $X_0 = X^{irr}(M)(u, v)$ and $X_1 = X^{red}(M)$ (see Exercise 5.8(2)). It is known that with respect to meridian-longitude coordinates,

$$A_{X_0}(u, v) = -v + vu^2 + u^4 + 2u^4v + v^2u^4 + vu^6 - vu^8$$

([CCGLS, Appendix]), whose Newton polygon is depicted in Figure 5.

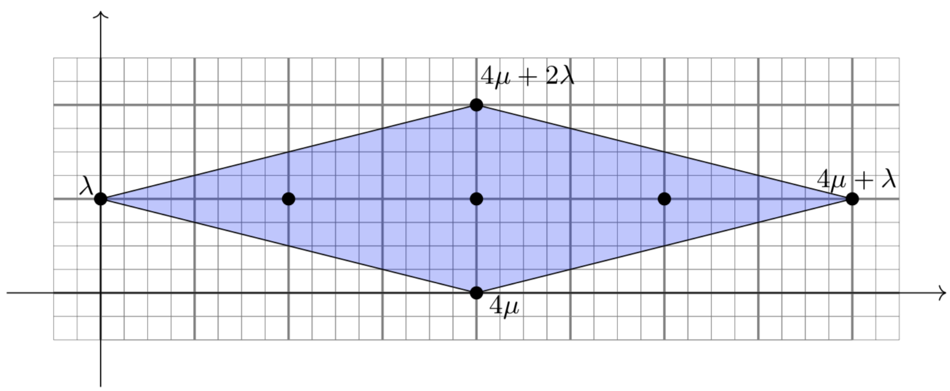

FIGURE 5. The Newton polygon of the figure eight knot

(The dots are the lattice points (n, m) with $a_{n,m} \neq 0$.) The reader will verify that N_{X_0} is dual to the Culler-Shalen norm ball of X_0 described in Example 7.11.

(2) A computer-aided calculation shows that the A-polynomial of the exterior M of the figure eight sister knot is

$$A_M(u, v) = 1 + (u^2 - u^4)v - 2u^4v^2 + (-u^4 + u^6)v^3 + u^8v^4$$

and therefore it's Newton polygon is as depicted in Figure 6, which is easily seen to be dual to its Culler-Shalen norm ball (see Example 9.2).

Sketch of the proof of the finite surgery theorem. Suppose that M is a hyperbolic knot manifold for which the statement of Theorem 9.1 does not hold. Let X_0 be a canonical curve of M and consider a weighted version of its A-polynomial

$$A^*_{X_0} = A^k_{X_0} \in \mathbb{C}[H_1(\partial M)]$$

where $k = \text{degree}(i^\# : X_0 \to i^\#(X_0))$. It is shown in [BZ3] how to extend X_0 to a finite collection of curves $X_0, X_1, \ldots, X_n$ satisfying

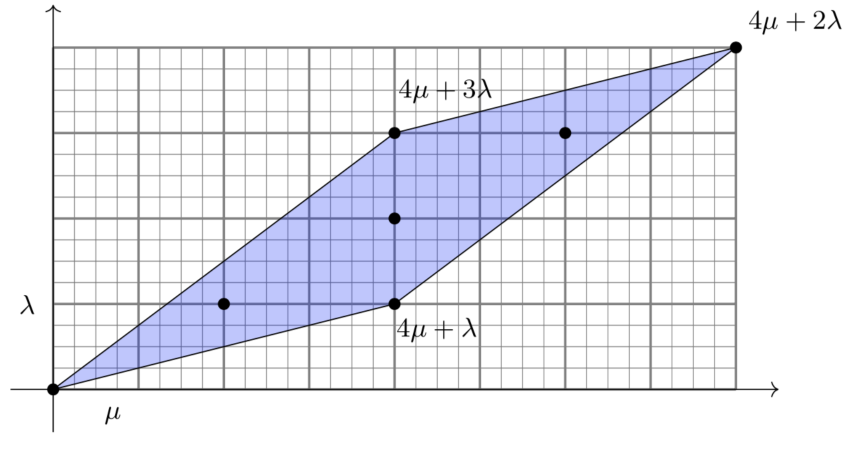

FIGURE 6. The Newton polygon of the figure eight sister knot

- degree$(i^{\#}|_{X_i})$ = degree$(i^{\#}|_{X_0})$ and $\|\cdot\|_{X_i} = \|\cdot\|_{X_0}$ for $i = 1, 2, \ldots, n$;
- $A^* = \prod_{i=1}^k A^*_{X_i} \in \mathbb{Z}[H_1(\partial M)]$ and the edge polynomials of A^* are products of cyclotomic polynomials;
- if $X = \cup_i X_i$ then the norm balls of $\|\cdot\|_X = \sum_i \|\cdot\|_{X_i}$ are geometrically dual to the Newton polygon N_X of A_X.

The constraints on $\|\alpha\|_{X_0}$ discussed in §9.1 are also satisfied by $\|\cdot\|_X$, and they limit the possible shapes of the norm balls of $\|\cdot\|_X$. Theorem 9.9 converts these restrictions to ones on the Newton polygon of A^* which, when combined with the conditions on the coefficients of A^*-polynomials described above, show that the only possibility is for $s_X = 8$, B_X to be a parallelogram, and there is a basis for $H_1(\partial M)$ and $\theta \in \{\pm 1\}$ for which

$$A_X(u, v) = u^3 + (\theta - u - \theta u^2 - \theta u^4 - u^5 + \theta u^6)v^2 + u^3 v^4$$

Work of Craig Hodgson shows that if the given polynomial was the A-polynomial of a hyperbolic 3-manifold, then the real 1-form

$$\omega = ln|u|d(arg(v)) - ln|v|d(arg(u))$$

is exact on the smooth part of $D^* = (A^*)^{-1}(0)$. (See [CCGLS, §4] for a proof of this fact.) The proof of Theorem 9.1 is completed by exhibiting a closed loop in D^* over which the integral of ω is non-zero. In other words, the A_X under consideration cannot come from a knot manifold. □

Remark 9.11. Though the techniques used to analyse the Culler-Shalen norms and A-polynomials of hyperbolic knot manifolds which admit multiple finite filling slopes also work for general small Seifert filling slopes, it is an open challenge to apply them to deduce strong bounds on the distance between such slopes. One problem is that though finite 3-manifold groups typically have very few irreducible $PSL(2, \mathbb{C})$-characters, the number of such characters of the fundamental groups of small Seifert manifolds cannot be a priori bounded ([BB]).

10. Dehn fillings with positive dimensional character varieties

Consider a knot manifold M, not necessarily hyperbolic, and a primitive class $\alpha \in H_1(\partial M)$ for which $X^{irr}(M(\alpha))$ is positive dimensional. We discussed in §7.4 how the Culler-Shalen seminorm of a curve $X_0 \subseteq X^{irr}(M(\alpha)) \subseteq X^{irr}(M)$ satisfies

$$\|\beta\|_{X_0} = s\Delta(\alpha, \beta)$$

for some integer $s \geq 0$ (Exercise 7.6). Hence if $s \neq 0$, then $\|\beta\|_{X_0}$ realises the minimal non-zero value of the restriction of $\|\cdot\|_{X_0}$ to $H_1(\partial M)$ if and only if $\Delta(\alpha, \beta) = 1$. The restrictions on the Culler-Shalen seminorms of finite filling classes described in §9, which hold for more general choices of X_0, then suggest that their distance to α is small. This is systematically investigated in [BZ2]. The following subsection provides a result illustrating this phenomenon.

10.1. **Seifert surgery on knots in the 3-sphere.** Let M be the exterior of a non-trivial knot K in the 3-sphere and $\mu, \lambda \in H_1(\partial M)$ denote a meridian, longitude pair for K (§8.2). The following conjecture of Cameron McA. Gordon ([Go2, Conjecture 4.8]) concerns when surgery on a knot in the 3-sphere can yield a Seifert fibre space.

Conjecture 10.1 (Cameron McA. Gordon)**.** *If α-surgery on a non-torus knot in the 3-sphere yields a Seifert manifold, then $\Delta(\alpha, \mu) \leq 1$. That is, α is either meridional or an integer slope.*

We can use character variety methods to deal with the case that the Seifert manifold is Haken.

Theorem 10.2. ([BZ2, Corollary 1.7]) *If M is the exterior of a non-trivial knot K in the 3-sphere and α a slope on ∂M for which $M(\alpha)$ is a Haken Seifert fibre space, then α is an integer slope.*

Proof. The proof in the case that K is a satellite knot requires some detailed arguments from classical 3-manifold topology (see [BZ2, Proof of Theorem 1.6]), so we restrict our attention to the case that M does not contain any essential tori in order to emphasize the role of character varieties.

Since $M(\alpha)$ is Haken, $\alpha \neq \pm\mu$, so it suffices to show that $\Delta(\alpha, \mu) = 1$. We assume, without loss of generality, that $\alpha \neq \pm\lambda$, as otherwise it would be an integer slope. Thus $H_1(M(\alpha))$ is a finite cyclic group.

The hypothesis that $M(\alpha)$ is Haken implies that it contains an essential surface S, and as it is a Seifert fibre space, S can be isotoped to be either horizontal or a vertical torus ([Ja, VI.34])). Our assumption that $H_1(M(\alpha))$ is a finite group implies that S must be separating. If it were horizontal, it would split $M(\alpha)$ into two twisted I-bundles ([Ja, VI.34]), which is impossible since this would imply that the cyclic group $H_1(M(\alpha))$ admits an epimorphism to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Thus S is a separating vertical torus.

If μ is a boundary slope and $\Delta(\alpha, \mu) > 1$, the intersection graph method of Gordon and Luecke (see [CGLS, Theorem 2.0.3]) implies that M contains a closed, essential surface which stays essential in $M(\alpha)$, and we can take this surface to be S. But then M would contain an essential torus, contrary to our assumptions. Thus $\Delta(\alpha, \mu) \leq 1$ when μ is a boundary slope.

Next assume that μ is not a boundary slope. We claim that the base orbifold of $M(\alpha)$ is hyperbolic. To see this, note that the vertical torus S splits $M(\alpha)$ into two Seifert fibre spaces M_1, M_2, each with incompressible boundary S, and this induces a splitting $\mathcal{B} = \mathcal{B}_1 \cup_C \mathcal{B}_2$ of the base orbifold $\mathcal{B}$ of $M(\alpha)$, where $\mathcal{B}_i$ is the base orbifold of M_i and $C = \partial \mathcal{B}_i$ is the projection of S. Write $\mathcal{B}_1 = B_1(p_1, \ldots, p_n)$, where

B_1 is a compact connected surface with boundary C and the $p_i \geq 2$ are the orders of the exceptional Seifert fibres of M_1 (see [Th, Chapter 13], [Sc, §3]).

The Euler characteristic of $\mathcal{B}_1$ is given by $\chi(B_1) - \sum_i (1 - \frac{1}{p_i})$, so is positive if and only if $\mathcal{B}_1$ is either D^2 or $D^2(p)$. But these possibilities are easily ruled out since M_1 would then be a solid torus and therefore S would be compressible.

If $\chi(\mathcal{B}_1) = 0$, then $\mathcal{B}_1$ is $D^2(2,2)$ or a Möbius band, in which case M_1 is a twisted I-bundle over the Klein bottle ([Ja, VI.5(d)]). A similar argument shows that if $\chi(\mathcal{B}_2) \geq 0$, then $\chi(\mathcal{B}_2) = 0$ and M_2 is a twisted I-bundle over the Klein bottle. We saw above that S cannot split $M(\alpha)$ into two twisted I-bundles, so $\chi(\mathcal{B}_i) < 0$ for at least one i. Then $\chi(\mathcal{B}) = \chi(\mathcal{B}_1) + \chi(\mathcal{B}_2) \leq \chi(\mathcal{B}_i) < 0$. It follows that $\mathcal{B}$ is hyperbolic ([Th, Chapter 13]). Example 4.2(2) can then be used to build a positive dimensional family of conjugacy classes of representations $\pi_1(M) \to PSL(2,\mathbb{C})$. As noted at the end of that example, these representations lift to $SL(2,\mathbb{C})$, so we can find a complex affine curve X_0 in $X^{irr}(M)$. Arguing as in §7.4, there is an integer $s \geq 0$ such that for each $\beta \in H_1(\partial M)$ we have

$$\|\beta\|_{X_0} = s|\beta \cdot \alpha|$$

We consider two cases.

- If $s \neq 0$, then $\|\mu\|_{X_0} = s|\mu \cdot \alpha| \geq s > 0$ and the proof of Proposition 8.7 can be modified to show that as a cyclic filling slope which is not a boundary slope, $\|\mu\|_{X_0}$ is minimal amongst all elements of $H_1(\partial M)$ which are rationally independent of α. But this minimum is clearly s, so $\Delta(\mu, \alpha) = |\mu \cdot \alpha| = 1$.
- If $s = 0$, we noted at the end of §7.4 that M contains a closed essential surface associated to an ideal point of X_0 which stays essential in $M(\beta)$ whenever $\Delta(\alpha, \beta) > 1$. As $M(\mu) = S^3$, we must have $\Delta(\mu, \alpha) \leq 1$.

□

10.2. **Curves of $PSL(2,\mathbb{C})$-characters.** To make the most of curves of characters coming from Dehn fillings of a knot manifold, it is best to work with the Culler-Shalen seminorms of curves contained in the $PSL(2,\mathbb{C})$-character variety of M. These are entirely analogous to their $SL(2,\mathbb{C})$ cousins in terms of construction, properties, and impact, and we refer the reader to [BZ2] for their development. They are also more generally applicable for the simple reason that there are knot manifolds M and curves in their $PSL(2,\mathbb{C})$-character varieties which do not lift to $X(M)$. The following exercise gives an idea of why this is.

Exercise 10.3. Show that there is an infinite family of irreducible representations $\mathbb{Z}/2 * \mathbb{Z}/3 \to PSL(2,\mathbb{C})$, but no such family of representations $\mathbb{Z}/2 * \mathbb{Z}/3 \to SL(2,\mathbb{C})$.

Example 10.4. Let M be a knot manifold which contains no closed essential surface and X_0 a curve of $PSL(2,\mathbb{C})$-characters of $M(\alpha)$ containing irreducible characters. Then

$$\|\beta\|_{X_0} = s|\alpha \cdot \beta|$$

where $s \neq 0$ (Lemma 7.4(1)). It is shown in [BZ2, Theorem 6.2] that if β is a finite filling slope then

$$s\Delta(\alpha,\beta) = \|\beta\|_{X_0} \leq \begin{cases} s & \text{if } \beta \text{ has } C\text{-type} \\ 2s & \text{if } \beta \text{ has } D\text{-type} \\ s+2 & \text{if } \beta \text{ has } T\text{-type} \\ s+3 & \text{if } \beta \text{ has } O\text{-type} \\ s+4 & \text{if } \beta \text{ has } I\text{-type} \end{cases}$$

Consequently,

$$\Delta(\alpha,\beta) \leq \begin{cases} 1 & \text{if } \beta \text{ has } C\text{-type} \\ 2 & \text{if } \beta \text{ has } D\text{-type} \\ 3 & \text{if } \beta \text{ has } T\text{-type} \\ 4 & \text{if } \beta \text{ has } O\text{-type} \\ 5 & \text{if } \beta \text{ has } I\text{-type} \end{cases}$$

Taking M to be the trefoil exterior shows that these inequalities are sharp. Indeed, $M(h) \cong L(3,2)\#L(2,1)$, where $h = 6\mu + \lambda$, so the $PSL(2,\mathbb{C})$-character variety of M contains that of $\mathbb{Z}/2 * \mathbb{Z}/3$. A calculation as in Example 5.7 shows that there is a unique curve X_0 in the latter and that $\|\beta\|_{X_0} = \Delta(\beta, h)$ (i.e., $s = 1$). For each slope $\beta \neq h$ on ∂M, $M(\beta)$ admits a Seifert structure with base orbifold $S^2(2,3,\Delta(\beta,h))$. Then $M(\beta)$ has C-type, D-type, T-type, O-type, or I-type if and only if $\Delta(\beta,h)$ is, respectively, $1,2,3,4$, or 5. Figure 7 below details the unit ball B of $\|\cdot\|_{X_0}$ and depicts the slopes of distance $1,2,3,4$, or 5 from h as, respectively, red dots, yellow diamonds, orange triangles, lime squares, or blue pentagons.

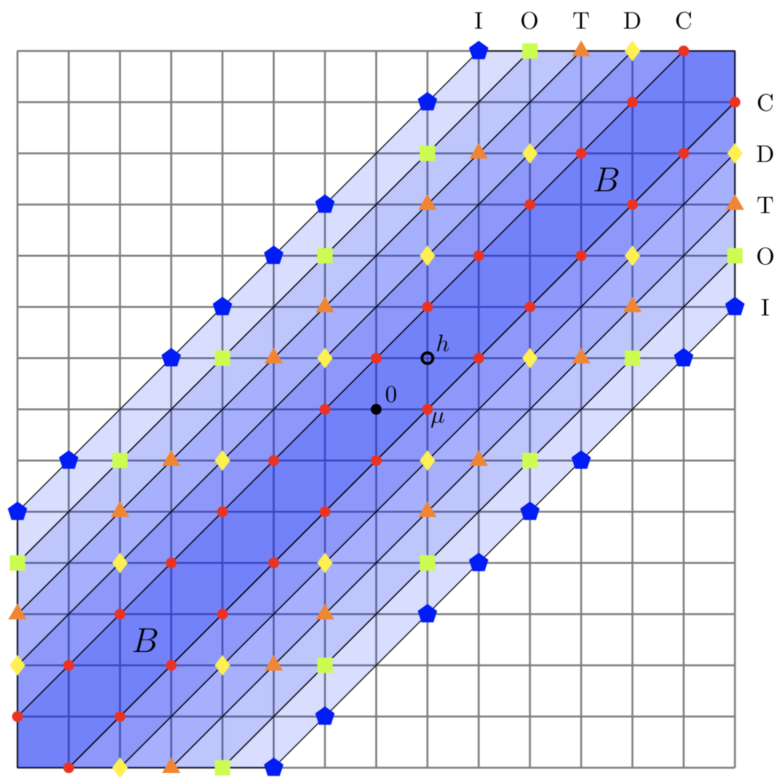

Figure 7. Finite filling slopes for the trefoil knot

More generally, it is shown in [BZ2] how to combine character variety methods with intersection graph techniques to obtain good bounds on distances between finite filling slopes and slopes yielding reducible or Seifert fibre spaces. Since most Seifert fibre spaces have $PSL(2,\mathbb{C})$-character varieties of dimension 2 or more (Example 4.2(2)), much more can be said in the latter case. For if a knot manifold M admits a Seifert fibre space Dehn filling $M(\alpha)$ whose character variety has dimension at least 2, there is a closed essential surface in M associated to an ideal point of some curve $X_0 \subset X_{PSL_2}(M(\alpha)) \subset X_{PSL_2}(M)$

which stays essential in $M(\beta)$ whenever $\Delta(\alpha,\beta) \geq 2$ (see [BZ2, Proposition 4.10] and [BZ2, Claim, page 486]). This leads to the following result.

Theorem 10.5. ([BGZ1, Theorems 1.5 and 1.7] *Let M be a compact, connected, orientable, hyperbolic 3-manifold whose boundary is a torus. Suppose that $M(\alpha)$ is a Seifert fibred space whose base orbifold is hyperbolic, though neither a 2-sphere with three cone points nor a projective plane with two cone points. If $\beta \in \mathcal{E}(M)$, then*

$$\Delta(\beta,\alpha) \leq \begin{cases} 1 & \text{if } M(\beta) \text{ is reducible or small} \\ 1 & \text{if } M(\beta) \text{ is small Seifert} \\ 3 & \text{if } M(\beta) \text{ is toroidal} \end{cases}$$

11. Some final remarks on distance bounds and the geography of exceptional fillings

We saw in §8.1 that an a priori upper bound on the distance between elements of a set of slopes $\mathcal{S}$ on the boundary of a hyperbolic knot manifold M leads to a bound on the size of $\mathcal{S}$ and to restrictions on their relative positioning in the surgery plane $H_1(\partial M;\mathbb{R})$. Thus, the geography of $\mathcal{E}(M)$ in $H_1(\partial M;\mathbb{R})$ is well-illustrated by a refined knowledge of the distance bounds amongst and between its subsets consisting of reducible filling slopes, of toroidal filling slopes, and of small Seifert filling slopes. We have seen, moreover, that the existence of such bounds has interesting topological consequences (e.g. Corollaries 8.5 and 8.6).

The intersection graph method of Gordon and Luecke yields sharp upper bounds in the case of slopes producing reducible or toroidal manifolds, as well as much about the topology of manifolds which realize the upper bounds. For instance,

- $\Delta(\alpha,\beta) \leq 1$ if α and β are reducible filling slopes ([GLu]);
- $\Delta(\alpha,\beta) \leq 3$ if α is a reducible filling slope and β is a toroidal filling slope ([Oh]);
- $\Delta(\alpha,\beta) \leq 8$ if α and β are toroidal filling slopes and the triples (M,α,β), where α and β are toroidal filling slopes with $\Delta(\alpha,\beta) \geq 4$, have been classified ([Go3], [GW]).

Each of these bounds is sharp in that there is a hyperbolic knot manifold M and slopes α,β such that $M(\alpha), M(\beta)$ are exceptional fillings of the listed types and $\Delta(\alpha,\beta)$ achieves the bound.

Understanding the relationship between small Seifert filling slopes and other exceptional slopes has proved challenging. For instance, small Seifert manifolds with finite first homology groups do not contain essential surfaces, so the intersection graph method is a non-starter. On the other hand, the generic small Seifert manifold does contain an essential immersed torus and this fact can be used to deduce the following results:

- $\Delta(\alpha,\beta) \leq 4$ if α is a small Seifert filling slope and β is a reducible filling slope which is also a strict boundary slope[4] (e.g., $M(\beta) \not\cong S^1 \times S^2, P^3 \# P^3$) ([BCSZ]);
- $\Delta(\alpha,\beta) \leq 5$ if α is a small Seifert filling slope, β is a toroidal filling slope which is also a strict boundary slope[5] (e.g., $M(\beta)$ is neither a torus bundle over the circle nor the union of two twisted I bundles over the Klein bottle), and M is not the figure eight knot exterior ([BGZ3], [BGZ4], [BGZ5]).

[4] Reducible filling slopes of a hyperbolic knot manifold are always boundary slopes.

[5] Toroidal filling slopes of a hyperbolic knot manifold are always boundary slopes.

The following table details the current state of knowledge concerning upper bounds on the distance between various families of exceptional filling slopes. Each entry lists the known upper bound on $\Delta(\alpha, \beta)$ for the corresponding types of exceptional filling classes. An entry annotated with an asterisk indicates that the bound is sharp. A slope annotated with a superscript "*str*" indicates that it is assumed to be a strict boundary slope.

α \ β	reducible	cyclic	finite	small Seifert	toroidal
reducible	1^* ([GLu])	1^* ([BZ2])	1^* ([BGZ2])	8 ([LM]) 4 (α^{str}, [BCSZ])	3^* ([Oh])
cyclic	·	1^* ([CGLS])	2^* ([BZ1])	8 ([LM])	4 ([Lee])
finite	·	·	3^* ([BZ3])	8 ([LM])	5^* (β^{str}, [BGZ5])
small Seifert	·	·	·	8 ([LM])	8 ([LM]) 7^* (β^{str}, [BGZ5])
toroidal	·	·	·	·	8^* ([Go3])

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