Virtual Haken 3-Manifolds and Dehn Filling

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§0. Introduction

The study of 3-manifolds splits nicely into the cases of finite fundamental groups and infinite fundamental groups. Concerning 3-manifolds with infinite fundamental groups, the following important conjecture due to Waldhausen [Wa] is well known.

Conjecture 0.1. Any closed, connected, orientable, irreducible 3-manifold W with infinite fundamental group is virtually Haken, i.e. W has a finite cover which is a Haken 3-manifold.

A stronger conjecture is that any closed, connected, orientable, irreducible 3-manifold W with infinite fundamental group has a virtually positive first Betti number, i.e. W has a finite cover which has positive first Betti number.

Conjecture 0.1 becomes more compelling due to the recent work of Gabai-Meyerhoff-Thurston [GMT]. In fact it follows from [GMT] (as well as [Ga1-2], [Th1], [CJ]) that if a closed 3-manifold W is virtually Haken, then W is topologically rigid and admits a geometric decomposition in Thurston's sense [Th1].

In this paper we consider the conjecture through the Dehn filling construction. Let M be a compact, connected, orientable, irreducible 3-manifold M such that ∂M is a torus. Recall that a slope on ∂M is the isotopy class of an unoriented, simple, essential loop in ∂M. We use $\Delta(r_1, r_2)$ to denote the *distance* (i.e. the minimal geometric intersection number) between two slopes r_1 and r_2 on ∂M and use $M(r)$ to denote the closed 3-manifold obtained by Dehn filling M along ∂M with slope r.

Call a slope r on ∂M a *virtually Haken filling slope* if $M(r)$ is a virtually Haken 3-manifold. According to Thurston [Th1], either M is a Seifert fibred manifold, or it contains an incompressible, non-boundary parallel torus, or it is hyperbolic, i.e. int(M) admits a complete hyperbolic structure of finite volume. In the first two cases there is quite a lot known about the virtually Haken filling slopes on ∂M [He2] so we shall concentrate, for the most part, on the case where M is hyperbolic. Here work of C. Gordon and J. Luecke [GL] and S. Boyer and X. Zhang [BZ1] show that there are no more than nine slopes on ∂M whose associated fillings are either reducible or have a finite fundamental group. Hence

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if Conjecture 0.1 holds, we are left with the striking conclusion that the set of slopes on ∂M which are not virtually Haken filling slopes has fewer than ten elements. A measure of the depth of the conjecture is that to date there has been little empirical evidence which supports this conclusion. For instance combined work of M. Baker [Ba], J. Hempel [He2], S. Kojima and D. Long [KL], and A. Nicas [N] has shown that roughly 70% of the fillings of the exterior of the figure 8 knot are virtually Haken manifolds, though the status of the remaining cases is open. In this paper we show that certain hypotheses on a filling $M(r_0)$ of M can be used to prove that if $\Delta(r, r_0)$ is large enough, then $M(r)$ is a virtually Haken manifold. This work is then combined with a recent result of D. Cooper and D. Long [CL] to construct manifolds M for which all but finitely many of the fillings are virtually Haken, but non-Haken manifolds. These appear to be the first such examples known.

By an *essential surface* in a compact orientable 3-manifold, we mean a properly embedded orientable surface each component of which is incompressible and non-boundary-parallel. A slope r on ∂M is called a *boundary slope* if there is a connected essential surface F in M such that ∂F is not empty and is of slope r. A *strict boundary slope* is a slope r for which there is a connected essential surface F in M satisfying

- ∂F is not empty and is of slope r;
- F is not a fibre of some fibration of M over the circle;
- F does not split M into two twisted I-bundles.

The *strict genus* of a strict boundary slope r is the minimal genus of all essential surfaces F in M satisfying the conditions above. The following theorem is proven in [BCSZ].

Theorem 0.2. (Boyer-Culler-Shalen-Zhang) *Let M be a compact, connected, orientable, irreducible 3-manifold such that ∂M is a torus and the interior of M admits a complete hyperbolic metric of finite volume. Suppose that r_0 is a strict boundary slope of strict genus g on ∂M. If r is any other slope on ∂M which satisfies $\Delta(r, r_0) > 20g + 5$, then $M(r)$ is irreducible and $\pi_1(M(r))$ contains a free subgroup of rank 2.* □

Conjecture 0.1 together with Theorem 0.2 suggest that the following holds.

Conjecture 0.3. For a compact, connected, orientable, irreducible 3-manifold M whose boundary is a torus and whose interior admits a complete hyperbolic metric of finite volume, the distance between a strict boundary slope of genus g and a non-virtually Haken filling slope on ∂M is less than or equal to $20g + 5$.

The condition that one of the slopes be a strict boundary slope can probably be replaced by the weaker condition that it be a boundary slope, though most results to date require the stronger condition as a hypothesis. Note that in the more general situation the hyperbolicity condition is necessary, as can be seen by taking M to be the exterior of a non-trivial torus knot. In this case the longitudinal slope on ∂M is a boundary slope, but not a strict one, and there are non-virtually Haken filling slopes on ∂M which are of arbitrarily large distance from it [Mo]. It is interesting to note that there is no universal bound, independent of genus, for the distance between a boundary slope and a non-virtually Haken filling slope, as is shown by examples of Bleiler and Hodgson [BH, Proposition 18].

Recently Cooper and Long [CL] have provided the following supporting evidence for the truth of Conjecture 0.3.

Theorem 0.4. (Cooper-Long [CL]) *Suppose that M is a compact, connected, orientable, irreducible 3-manifold such that ∂M is a torus and that the interior of M admits a complete hyperbolic metric of finite volume. Suppose that M does not fibre over the circle and that there is a connected essential surface F of genus g in M such that ∂F is connected of slope r_0 on ∂M. Then any slope r on ∂M satisfying $\Delta(r, r_0) \geq 12g - 4$ is a virtually Haken filling slope.* $\square$

We shall prove several theorems below which provide other supporting evidence for Conjecture 0.3. Our results are based on the following theorem and the methods developed in [BZ2].

Let G and H be groups. We say that G is *virtually H-representable* if G has a finite index subgroup which admits a homomorphism onto H.

Theorem 0.5. *Suppose that M is a compact, connected, orientable, irreducible 3-manifold whose boundary is a torus and that there is no closed essential surface in M. Suppose further that for some slope r_0 on ∂M, there is a surjective homomorphism $\phi : \pi_1(M(r_0)) \to \Gamma$ where Γ is the orbifold fundamental group of a 2-dimensional hyperbolic orbifold $\mathcal{B}$ of the form $B(p_1, p_2, \ldots, p_m) \neq S^2(p, q, r)$. Then if r is a slope on ∂M satisfying $\Delta(r, r_0) > 5$, $\pi_1(M(r))$ is virtually $\mathbf{Z}$-representable. Further if $\Delta(r, r_0) > 6$, then $\pi_1(M(r))$ is virtually $\mathbf{Z} * \mathbf{Z}$-representable.*

Theorem 0.5 can be refined as follows. Recall the *orbifold Euler characteristic* of a 2-dimensional orbifold of the form $B(p_1, p_2, \ldots, p_m)$ is given by

$$\chi^{orb}(\mathcal{B}) = \chi(B) - \sum_{j=1}^{m}(1 - 1/p_j) \tag{1}$$

where $\chi(B)$ is the usual Euler characteristic of the surface B.

Addendum. *Assume the conditions of Theorem 0.5. If r is a slope on ∂M such that $\chi^{orb}(\mathcal{B}) + \frac{1}{\Delta(r,r_0)} \leq 0$, then $\pi_1(M(r))$ is virtually $\mathbf{Z}$-representable, and if $\chi^{orb}(\mathcal{B}) + \frac{1}{\Delta(r,r_0)} < 0$, then $\pi_1(M(r))$ is virtually $\mathbf{Z} * \mathbf{Z}$-representable.*

Remarks. (1) The condition that M be *small*, i.e. it contains no closed, essential surface, gives rise to what is in some ways the most interesting case for investigation. On the one hand all but finitely many fillings of a small manifold are non-Haken ([Ha]), so the virtual Haken nature of these manifolds is subtle. On the other hand if M contains a closed, essential surface S, there is always a slope r_0 on ∂M such that S is incompressible in $M(r)$ for any slope r satisfying $\Delta(r, r_0) > 1$ ([Wu1]). Since at most three fillings of M are reducible [GL], the generic filling of M is Haken. Of course there is in general no a priori way to determine such an r_0, and so identify the slopes which yield Haken manifolds. Hence it is still of interest to study the fillings of manifolds which are not small.

(2) The conditions of Theorem 0.5 imply that the slope r_0 is a strict boundary slope (see the proof of Theorem 0.5).

(3) If the fundamental group of a 3-manifold W is virtually $\mathbf{Z}$-representable, then its virtual first Betti number $b_1^{virt}(W) = \sup\{\text{rank}(H_1(\tilde{W})) \mid \text{there is a finite cover } \tilde{W} \to W\}$ is positive.

(4) If the fundamental group of a 3-manifold W is virtually $\mathbf{Z} * \mathbf{Z}$-representable, then it is virtually H-representable for an arbitrary finitely generated group H. Hence $b_1^{virt}(W) = \infty$.

In Theorems 0.6 and 0.7 below we show that the requirement in Theorem 0.5 that M be small can be removed in various situations (see Remark 1 above).

Theorem 0.6. *Suppose that M is a compact, connected, orientable, 3-manifold whose boundary is a torus and whose interior has a hyperbolic metric of finite volume. Suppose that r_0 is a slope on ∂M such that $M(r_0)$ is a reducible manifold which is not $S^1 \times S^2$ or $RP^3 \# RP^3$.*

(i) If $M(r)$ is not virtually Haken, then $\Delta(r, r_0) \leq 5$.

(ii) If $\pi_1(M(r))$ does not contain a free group of rank 2, then $\Delta(r, r_0) \leq 6$.

(iii) If we assume further that $M(r_0)$ is not a connected two lens spaces of orders p, q where $1/p + 1/q < 1/2$, then the distance bounds given in parts (i) and (ii) can be reduced to 1.

Remark. The conditions of Theorem 0.6 imply that the slope r_0 is a strict boundary slope of genus zero. On the other hand if we add the hypothesis that r_0 be a strict boundary slope, then the condition that $M(r_0) \neq S^1 \times S^2$ can be removed.

Theorem 0.7. *Suppose that M is a compact, connected, orientable 3-manifold whose boundary is a torus and whose interior has a hyperbolic metric of finite volume. Suppose that for some slope r_0 on ∂M, $M(r_0)$ admits the structure of a Seifert fibred space whose base orbifold $\mathcal{B}$ is hyperbolic and is not of the form $S^2(p, q, r)$. If r is a slope on ∂M such that $\Delta(r, r_0) > 5$, then $M(r)$ is virtually Haken.*

Remarks. (1) The theorem can be refined: if r is a slope on ∂M such that $\chi^{orb}(\mathcal{B}) + 1/\Delta(r, r_0) \leq 0$, then a finite cover of $M(r)$ contains an essential surface.

(2) A closed Seifert fibred manifold whose base orbifold is hyperbolic and does not have the form $S^2(p,q,r)$ contains an essential torus. Therefore as M is assumed to be hyperbolic, r_0 is necessarily a strict boundary slope on ∂M of genus less than or equal to 1.

One of the key ingredients in the proofs of Theorems 0.5, 0.6, and 0.7 is a general result, essentially due to Baumslag, Morgan, and Shalen, on the virtual **Z**-representability and **Z**$*$**Z**-representability of certain quotients of a Fuchsian group Γ [BMS] (though they considered a special case, their proof applies to a more general situation). This is the focus of the next section of the paper, where we prove Theorem 1.2 from which Theorem 0.5 will follow directly. The proof of Theorems 0.6 and 0.7 will depend on Theorem 0.5 and some results obtained in [BZ2] concerning the incompressibility, after Dehn filling, of certain essential closed surfaces in M associated to the character varieties of M. The proofs of Theorems 0.5, 0.6, and 0.7 and the Addendum to Theorem 0.5 will be given in §2. In §3 we give some examples of 3-manifolds M which satisfy the hypotheses of Theorem 0.5 and Theorem 0.6. In particular, we will give an infinite family of small knots in S^3 (i.e. knots whose exteriors are small manifolds) such that each one has a hyperbolic exterior M and except for at most finitely many slopes, all other slopes on ∂M yield manifolds which are non-Haken but virtually Haken. Such examples of small knots seem to be the first known.

We work in the smooth category and use the standard 3-manifold and knot theory terminology as found, for instance, in [He1] and [R].

§1. Virtual representations of $\Gamma / \ll \gamma^n \gg$

Throughout this section Γ will denote the orbifold fundamental group of a 2-dimensional hyperbolic orbifold $\mathcal{B}$ of the form $B(p_1, p_2, \ldots, p_m)$ where $2 \leq p_1, p_2, \ldots, p_m$. It is known that the hyperbolicity of $\mathcal{B}$ implies that

$$\chi(\Gamma)(= \chi^{orb}(\mathcal{B})) < 0. \qquad (2)$$

Let $\gamma_1, \gamma_2, \ldots \gamma_m \in \Gamma$ be elements of order $p_1, p_2, \ldots, p_m$ corresponding to small loops about the cone points of $\mathcal{B}$.

Lemma 1.1. *A torsion element γ of Γ is conjugate to a power of some γ_i.*

Proof. The group Γ admits a properly discontinous action on $\mathbf{H}^2$ with quotient orbifold $\mathcal{B}$. The point stabilizers of this action are finite subgroups of Γ and if the stabilizer of a point p is non-trivial, then p maps to one of the cone points of $\mathcal{B}$. Hence each point stabilizer conjugates into a subgroup generated by some γ_i. Now it is well known that any $\gamma \in \Gamma$ which has finite order has a fixed point in $\mathbf{H}^2$ ([Be, Corollary, pg. 70]), and so in particular lies in some point stabilizer. The result follows. $\square$

For an element $\gamma \in \Gamma$, we use $\ll \gamma \gg$ to denote the normal subgroup of Γ generated by γ. Recall that the *deficiency* of a presentation of a group is the integer equal to the number of generators minus the number of relators in the presentation.

The method of proof of the following theorem is taken more or less verbatim from the proof of Theorem B of [BMS] (those authors considered the special case $\Gamma = \mathbf{Z}/p * \mathbf{Z}/q$).

Theorem 1.2. *Let $\gamma \in \Gamma$ and suppose that for some $n \geq 1$ there is a representation $\rho : \Gamma \to PSL_2(\mathbf{C})$ for which each $\rho(\gamma_i)$ has order p_i and $\rho(\gamma)$ has order n. Then there is an integer $d \geq 1$ and a finite index normal subgroup of $\Gamma / \ll \gamma^n \gg$ which admits a presentation $\mathcal{P}$ of deficiency*

$$\mathit{def}\ (\mathcal{P}) = 1 - d(\chi(\Gamma) + 1/n).$$

Proof. As $\rho(\Gamma)$ is residually finite (being a finitely generated subgroup of $PSL_2(\mathbf{C}) \subset SL_3(\mathbf{C})$), there is a normal subgroup H of finite index in $\rho(\Gamma)$ such that each $\rho(\gamma_i)$ is of order p_i modulo H and $\rho(\gamma)$ has order n modulo H. Set $\tilde{\Gamma} = \rho^{-1}(H)$ and observe that

$$\Gamma / \tilde{\Gamma} \cong \rho(\Gamma)/H.$$

In particular $\tilde{\Gamma}$ has some finite index, say d, in Γ.

We claim that $\tilde{\Gamma}$ is torsion free. For by Lemma 1.1, if $\tilde{\gamma} \in \tilde{\Gamma}$ and there is an integer $k \geq 1$ such that $\tilde{\gamma}^k = 1$, then $\tilde{\gamma}$ is conjugate to a power of some γ_i, say $\tilde{\gamma}$ is conjugate to γ_i^l where $l \geq 1$. But then $\gamma_i^l \in \tilde{\Gamma}$ so that $\rho(\gamma_i^l) \in H$. Thus p_i, the order of $\rho(\gamma_i)$ modulo H, divides l. Hence $\gamma_i^l = 1$ and so $\tilde{\gamma} = 1$, i.e. $\tilde{\Gamma}$ is torsion free. It follows that $\tilde{\Gamma}$ is the fundamental group of a compact, connected Euclidean or hyperbolic surface. It is easy to check that the standard presentation $\mathcal{P}_0$ of $\tilde{\Gamma}$ has deficiency given by

$$\text{def } (\mathcal{P}_0) = 1 - \chi(\tilde{\Gamma}) = 1 - d\chi(\Gamma). \tag{3}$$

Now we know that γ has order n modulo $\tilde{\Gamma}$ and so Corollary 3 of [BMS] says that there is a presentation $\mathcal{P}$ of $\tilde{\Gamma} / \ll \gamma^n \gg$ obtained from $\mathcal{P}_0$ by adding d/n relators. It follows then from Equation (2) that $\mathcal{P}$ has deficiency def $(\mathcal{P}) =$ def $(\mathcal{P}_0) - d/n = 1 - d\chi(\Gamma) - d/n = 1 - d(\chi(\Gamma) + 1/n)$. $\square$

Corollary 1.3. *If $\chi(\Gamma) + 1/n \leq 0$, then $\Gamma / \ll \gamma^n \gg$ is virtually $\mathbf{Z}$-representable. If $\chi(\Gamma) + 1/n < 0$, then $\Gamma / \ll \gamma^n \gg$ is virtually $\mathbf{Z} * \mathbf{Z}$-representable.*

Proof. Groups having a presentation of deficiency 1 are virtually **Z**-representable, while those admitting a presentation of deficiency at least 2 are virtually $\mathbf{Z} * \mathbf{Z}$-representable [BP].

$\square$

It is a simple matter to determine for which pairs (Γ, n) the conditions of Corollary 1.3 are satisfied. Observe that Equation (1) implies that the following inequality holds

$$\chi(B) - m \leq \chi(\Gamma) = \chi(B) - \sum_{j=1}^{m}(1 - 1/p_j) \leq \chi(B) - m/2. \tag{4}$$

Recall that $\chi(\Gamma) < 0$.

Case 1. $\chi(B) < -1$.

In this case Inequality (4) shows that for each $n \geq 1$, $\chi(\Gamma) + 1/n < 0$, so that $\Gamma / \ll \gamma^n \gg$ is virtually $\mathbf{Z} * \mathbf{Z}$-representable.

Case 2. $\chi(B) = -1$.

Here $\Gamma / \ll \gamma^n \gg$ is always virtually **Z**-representable and is $\Gamma / \ll \gamma^n \gg$ is virtually $\mathbf{Z} * \mathbf{Z}$-representable unless $n = 1$ and $m = 0$.

Case 3. $\chi(B) = 0$.

In this case our assumption that $\chi(\Gamma) < 0$ gives $m \geq 1$ (Equation (1)). It then follows from Inequality (4) that

- when $m > 1$, $\Gamma / \ll \gamma^n \gg$ is virtually **Z**-representable for each $n \geq 1$, and is virtually $\mathbf{Z} * \mathbf{Z}$-representable unless $n = 1, m = 2$, and $p_1 = p_2 = 2$.
- when $m = 1$, $\Gamma / \ll \gamma^n \gg$ is virtually **Z**-representable for each $n \geq 2$, and is virtually $\mathbf{Z} * \mathbf{Z}$-representable for each $n \geq 2$ unless, perhaps, $n = p_1 = 2$.

Case 4. $\chi(B) = 1$.

Here the assumption that $\chi(\Gamma) < 0$ gives $m \geq 2$ and thus

- $\Gamma / \ll \gamma^n \gg$ is virtually **Z**-representable unless, perhaps, (i) $m = 3, n = 1$ and (p_1, p_2, p_3) is a Platonic triple (i.e. $1/p_1 + 1/p_2 + 1/p_3 > 1$), or (ii) $m = 2$ and (p_1, p_2, n) is a Platonic triple. Note $\chi(\Gamma) < 0$ precludes the possibility that $p_1 = p_2 = 2$.
- $\Gamma / \ll \gamma^n \gg$ is virtually $\mathbf{Z} * \mathbf{Z}$-representable unless, perhaps, (i) $m = 4, n = 1$, and $p_1 = p_2 = p_3 = p_4 = 2$, or (ii) $m = 3$, $n = 1$, and (p_1, p_2, p_3) is a Platonic triple, or (iii) $m = 3$, $n = 2$, and $p_1 = p_2 = p_3 = 2$, or (iv) $m = 2$ and (p_1, p_2, n) is a Euclidean or Platonic triple (i.e. $1/p_1 + 1/p_2 + 1/n \geq 1$). Note $\chi(\Gamma) < 0$ precludes the possibility that $p_1 = p_2 = 2$.

Case 5. $\chi(B) = 2$.

In this case the assumption that $\chi(\Gamma) < 0$ gives $m \geq 3$. Then

- if $m > 6$ then $\Gamma / \ll \gamma^n \gg$ is virtually $\mathbf{Z} * \mathbf{Z}$-representable for all $n \geq 1$.
- if $m = 6$ then $\Gamma / \ll \gamma^n \gg$ is virtually $\mathbf{Z}$ representable for all $n \geq 1$ and virtually $\mathbf{Z} * \mathbf{Z}$-representable unless $n = 1$ and $p_1 = p_2 = \ldots = p_6 = 2$.
- if $m = 5$ then $\Gamma / \ll \gamma^n \gg$ is virtually $\mathbf{Z} * \mathbf{Z}$-representable for all $n \geq 3$ and is virtually $\mathbf{Z}$-representable for all $n \geq 2$.
- if $m = 4$ then since $\chi(\Gamma) < 0$ we have $p_4 \geq 3$ and so $\Gamma / \ll \gamma^n \gg$ is virtually $\mathbf{Z} * \mathbf{Z}$-representable for all $n \geq 7$ and is virtually $\mathbf{Z}$-representable for all $n \geq 6$.
- if $m = 3$, then $\Gamma / \ll \gamma^n \gg$ is virtually $\mathbf{Z} * \mathbf{Z}$-representable for all (p_1, p_2, p_3) such that $1/p_1 + 1/p_2 + 1/p_3 + 1/n < 1$ and is virtually $\mathbf{Z}$-representable for all (p_1, p_2, p_3) such that $1/p_1 + 1/p_2 + 1/p_3 + 1/n \leq 1$.

As a consequence of these calculations we obtain the following result.

Corollary 1.4. *Suppose that $\mathcal{B}$ is not of the form $S^2(p, q, r)$, $\gamma \in \Gamma$, and for some n there is a representation $\rho : \Gamma \to PSL_2(\mathbf{C})$ for which each $\rho(\gamma_i)$ has order p_i and $\rho(\gamma)$ has order n. Then if $n > 5$, $\Gamma / \ll \gamma^n \gg$ is virtually $\mathbf{Z}$-representable, and if $n > 6$, $\Gamma / \ll \gamma^n \gg$ is virtually $\mathbf{Z} * \mathbf{Z}$-representable.*

§2. Proofs of Theorems 0.5, 0.6, and 0.7

First of all we must set the notation to be used and recall some of the results concerning the relations between the topology of 3-manifolds and the $PSL_2(\mathbf{C})$-character varieties of 3-manifolds. The reader is referred to [BZ2] for more details.

For any finitely generated group G, we use $\bar{R}(G) = Hom(G, PSL_2(\mathbf{C}))$ to denote the $PSL_2(\mathbf{C})$-representation variety of G and use $\bar{X}(G)$ to denote the algebro-geometric quotient of $\bar{R}(G)$ under the natural $PSL_2(\mathbf{C})$-action. The complex, affine, algebraic set $\bar{X}(G)$ is called the $PSL_2(\mathbf{C})$-*character variety* of G. There is a surjective, regular quotient map $\bar{t} : \bar{R}(G) \to \bar{X}(G)$. For a compact 3-manifold W, we use $\bar{R}(W)$ and $\bar{X}(W)$ to denote $\bar{R}(\pi_1(W))$ and $\bar{X}(\pi_1(W))$ respectively.

If $\bar{X}(G)$ is positive dimensional and X_0 be a curve in $\bar{X}(G)$, let $\hat{X}_0$ be a projective completion of X_0. Each point of $\hat{X}_0 \setminus X_0$ is called an *ideal point* of X_0.

For each $\gamma \in G$, the function $f_\gamma : \bar{X}(G) \rightarrow \mathbf{C}$ is defined by $f_\gamma(x) = [\mathrm{trace}(\widetilde{\rho(\gamma)})]^2 - 4$, where $\rho \in \bar{t}^{-1}(x) \subset \bar{R}(G)$ and $\widetilde{\rho(\gamma)}$ is an element of $\Phi^{-1}(\rho(\gamma))$ under the canonical map $\Phi : SL_2(\mathbf{C}) \rightarrow PSL_2(\mathbf{C})$. Obviously f_γ is well defined and it can be shown to be a regular function.

Let M be a compact, connected, orientable, irreducible 3-manifold with ∂M a torus. There is an injective homomorphism, well-defined up to conjugation, $\psi : H_1(\partial M) \equiv \pi_1(\partial M) \rightarrow \pi_1(M)$. Hence for each $\alpha \in H_1(\partial M)$ we may unambiguously define $f_\alpha = f_{\psi(\alpha)}$. To each slope r on ∂M we may associate a class $\alpha(r) \in H_1(\partial M)$, well-defined up to sign, by orienting one of its representative loops. We shall use the symbol f_r to denote the function $f_{\alpha(r)}$.

Suppose that $\bar{X}(M)$ is positive dimensional and suppose that $X_0 \subset \bar{X}(M)$ is a curve which contains the character of an irreducible representation. There is a unique 4-dimensional variety $R_0 \subset \bar{R}(M)$ such that $\bar{t}(R_0) = X_0$ ([BZ2, Lemma 4.1]). The following results from [BZ2] will be needed in the proofs of Theorems 0.5, 0.6, and 0.7.

(2.1) On X_0, either f_r is constant for each slope r, or there is a unique slope r_0 for which f_{r_0} is constant, or f_r is non-constant for each slope r [BZ2, §5].

(2.2) If at an ideal point of X_0, f_r has finite limiting value for each slope r, then there is a closed essential surface in M [BZ2, Proposition 4.7].

(2.3) If r_0 is the unique slope with f_{r_0} being constant on X_0, then r_0 is a boundary slope. Further if X_0 contains the character of a representation in R_0 whose image in $PSL_2(\mathbf{C})$ has no index two abelian subgroup, then r_0 is a strict boundary slope [BZ2, Proposition 4.7].

(2.4) If $\rho(r_0) = \pm I$ for each $\rho \in R_0$ and if at an ideal point of X_0, f_r has finite limiting value for each slope r, then there is a closed essential surface S in M such that if S compresses in both $M(r_0)$ and $M(r)$ for some slope r, then $\Delta(r, r_0) \leq 1$ [BZ2, Proposition 4.10].

Proof of Theorem 0.5 and its Addendum. The homomorphism ϕ induces an inclusion $\bar{X}(\Gamma) \subset \bar{X}(M(r_0))$. As we have assumed that $B(p_1, p_2, \ldots, p_m)$ is hyperbolic and not of the form $S^2(p, q, r)$, $\bar{X}(\Gamma)$ is positive dimensional and we may choose a curve $X_0 \subset \bar{X}(\Gamma) \subset \bar{X}(M(r_0)) \subset \bar{X}(M)$ which contains the character of a discrete faithful representation $\rho_0 : \Gamma \rightarrow PSL_2(\mathbf{R}) \subset PSL_2(\mathbf{C})$. Note that ρ_0 is an irreducible representation and its image in $PSL_2(\mathbf{C})$ has no finite-index abelian subgroup. By construction, f_{r_0} is constantly zero on X_0 and so by (2.2) above, the smallness of M implies that f_r blows up at each ideal point of X_0 for each slope $r \neq r_0$. In particular f_r is non-constant if $r \neq r_0$. By (2.3), r_0 is a strict boundary slope.

If $\gamma \in \Gamma$ is of finite order and $\rho \in \mathrm{Hom}(\Gamma, PSL_2(\mathbf{C}))$, then $\mathrm{trace}(\widetilde{\rho(\gamma)})$ is of the form $\pm 2\cos(2\pi j/2n)$ for some integers $j, n \geq 1$. In particular the functions f_{γ_i} are constant on

X_0 for $i = 1, ..., m$. Since we have chosen X_0 to contain the character of ρ_0, it can be shown that $f_{\gamma_i} \equiv 4((\cos(2\pi j_i/2p_i))^2 - 1)$ on X_0 for some integer j_i relatively prime to $2p_i$. Thus $\rho(\gamma_i)$ is of order p_i for each $\rho \in \bar{t}^{-1}(X_0)$ and for each of $i = 1, ..., m$.

Let $n \geq 2$ be fixed and consider a slope r on ∂M such that $\Delta(r, r_0) = n$. Fix a dual slope r_1 to r_0, i.e. a slope which satisfies $\Delta(r_1, r_0) = 1$. Since f_{r_1} is non-constant on X_0 and blows up at each ideal point of X_0, there is some point $x_1 \in X_0$ such that $f_{r_1}(x_1) = 4(((\cos(2\pi/2n))^2 - 1)$. It follows that if $\rho_1 \in \bar{t}^{-1}(x_1)$, then $\rho_1(r_1)$ has order n in $PSL_2(\mathbf{C})$.

Now by construction, ρ_1 factors as

$$\pi_1(M) \rightarrow \pi_1(M(r_0)) \xrightarrow{\phi} \Gamma \xrightarrow{\rho} PSL_2(\mathbf{C})$$

where ρ is a representation of Γ. If we let γ be any element of Γ obtained by orienting a representative loop for r_1 and attaching the result to the base point by some path, then $\rho(\gamma)$ has order n in $PSL_2(\mathbf{C})$. In the previous paragraph we observed that $\rho(\gamma_i)$ is of order p_i for each of $i = 1, ..., m$. Applying Corollary 1.4 to ρ implies that $\Gamma/ \ll \gamma^n \gg$ is virtually $\mathbf{Z}$ (respectively $\mathbf{Z} * \mathbf{Z}$)-representable as long as $n > 5$ (respectively $n > 6$).

The identity $\Delta(r, r_0) = n$ implies that orientations for the slopes may be chosen so that $\alpha(r) = s\alpha(r_0) + n\alpha(r_1)$ for some $s \in \mathbf{Z}$. It follows that the composition $\pi_1(M) \rightarrow \pi_1(M(r_0)) \xrightarrow{\phi} \Gamma$ induces a surjection $\psi : \pi_1(M(r)) \rightarrow \Gamma/ \ll \gamma^n \gg$. Thus $\pi_1(M(r))$ is virtually $\mathbf{Z}$ (respectively $\mathbf{Z} * \mathbf{Z}$)-representable as long as $n \geq 6$ (respectively $n > 7$). This completes the proof of Theorem 0.5. If we apply Corollary 1.3 instead of Corollary 1.4, we see that the Addendum of Theorem 0.5 holds. □

Proof of Theorem 0.6. If the first Betti number of M is larger than 1, then $M(r)$ has a positive first Betti number for each slope r on ∂M. According to [Ga3, Corollary], $M(r)$ is irreducible for any $r \neq r_0$ and by [Wu3, Theorem 4.1] we may assume that $M(r)$ does not contain an incompressible torus when $\Delta(r, r_0) > 1$. Hence if $\Delta(r, r_0) > 1$ then $M(r)$ is Haken and contains an incompressible surface of genus larger than 1. It therefore contains a free subgroup of rank 2.

Assume now that the first Betti number of M is 1. The reducibility theorem of Gordon and Luecke [GL] implies that $M(r)$ is irreducible as long as $\Delta(r, r_0) \geq 2$. Since r_0 is a boundary slope, it follows from [CGLS, Theorem 2.0.3] that one of the following three possibilities occurs.

(1) $M(r_0)$ is a connected sum of two non-trivial lens spaces (here non-trivial means different from $S^2 \times S^1$ and S^3); or

(2) M contains a closed essential surface S (of genus larger than 1 since M is hyperbolic)

which remains incompressible in $M(r)$ whenever $\Delta(r_0, r) > 1$; or

(3) $M(r) = S^2 \times S^1$.

By the hypotheses of Theorem 0.6 we may assume that the first possibility arises, though with at least one of the two nontrivial lens spaces different from the projective 3-space RP^3. Therefore $\pi_1(M(r_0)) \cong \mathbf{Z}/p * \mathbf{Z}/q$ with $\max\{p, q\} > 2$. Let ϕ denote any isomorphism $\pi_1(M(r_0)) \cong \mathbf{Z}/p * \mathbf{Z}/q$ and observe that $\mathbf{Z}/p * \mathbf{Z}/q \cong \Gamma = \pi_1^{orb}(D^2(p, q))$. Further $\chi^{orb}(D^2(p, q)) = -1 + (1/p + 1/q) \leq -1 + (1/2 + 1/3) = -1/6 < 0$.

We proceed now as in the proof of Theorem 0.5. Choose a curve $X_0 \subset \bar{X}(\Gamma) \subset \bar{X}(M(r_0)) \subset \bar{X}(M)$ which contains the character of a discrete faithful representation $\rho_0 : \mathbf{Z}/p * \mathbf{Z}/q \to PSL_2(\mathbf{C})$. By construction $f_{r_0}(x) = 0$ for each ideal point x of X_0. If for any slope $r \neq r_0$ we have $f_r(x) = \infty$ for each ideal point x of X_0, then we continue as in the proof of Theorem 0.5 to see that the parts (i) and (ii) of Theorem 0.6 hold. Further if $1/p + 1/q < 1/2$ then $\chi^{orb}(D^2(p, q)) + 1/\Delta(r, r_0) < 0$ as long as $\Delta(r, r_0) \geq 2$. Hence Corollary 1.3 implies that part (iii) of the theorem also holds.

Assume now that there is some ideal point x of X_0 and slope $r \neq r_0$ such that f_r has a finite limiting value at x. Then by (2.4), there is a closed essential surface S in M such that if r is any slope on ∂M for which $\Delta(r, r_0) > 1$, then S remains incompressible in one of $M(r_0)$ or $M(r)$. But clearly S must compress in $M(r_0)$, a connected sum of non-trivial lens spaces, and therefore S remains incompressible in $M(r)$ as long as $\Delta(r, r_0) \geq 2$. Thus Theorem 0.6 holds in this final case. □

Proof of Theorem 0.7. Let ϕ denote the natural surjection from $\pi_1(M(r_0))$ to $\Gamma = \pi_1^{orb}(\mathcal{B})$ and choose a curve $X_0 \subset \bar{X}(\Gamma) \subset \bar{X}(M(r_0)) \subset \bar{X}(M)$ which contains the character of a discrete faithful representation $\rho_0 : \Gamma \to PSL_2(\mathbf{C})$. By construction $f_{r_0}(x) = 0$ for each ideal point x of X_0. If for any slope $r \neq r_0$ we have $f_r(x) = \infty$ at each ideal point x of X_0, then we continue as in the proof of Theorem 0.5 to see that Theorem 0.7 holds.

Assume now that there is some ideal point x of X_0 and slope $r \neq r_0$ such that $f_r(x) \in \mathbf{C}$. By (2.4) there is a closed, connected, essential surface S in M such that if S compresses in both $M(r_0)$ and $M(r)$, then $\Delta(r, r_0) \leq 1$. Note that $\pi_1(S)$ lies in an edge stabilizer of the action of $\pi_1(M)$ on some tree associated to the ideal point x (see [BZ2, §4]). We claim that S must compress in $M(r_0)$. To see this, suppose otherwise. Since $M(r_0)$ is Seifert fibred, the surface S is isotopic in $M(r_0)$ to a vertical surface (i.e. consists of fibres) or a horizontal surface (i.e. intersects transversely to each fibre). But S cannot be vertical, as in this case it would be an essential torus in M, contradicting the fact that M is hyperbolic. Thus S must be horizontal and so is either the fibre of a realization of $M(r_0)$ as the total space of locally trivial bundle over the circle, or it splits $M(r_0)$ into two twisted I-bundles ([Ja, Lemma

VI.34]). Hence $\pi_1(S)$ is a normal subgroup of $\pi_1(M(r_0))$ and $\pi_1(M(r_0))/\pi_1(S)$ is either $\mathbf{Z}$ or $\mathbf{Z}/2 * \mathbf{Z}/2$. Let $P : \pi_1(M) \to PSL_2(\mathbf{C}(R_0))$ be the tautological representation ([BZ2, §4]) where $R_0 \subseteq \bar{t}^{-1}(X_0)$ is the unique 4-dimensional variety satisfying $\bar{t}(R_0) = X_0$ ([BZ2, Lemma 4.1]) and $\mathbf{C}(R_0)$ is the function field over R_0. By construction P factors through Γ.

Now according to Proposition 4.4 of [BZ2], either $P(\pi_1(S)) = \{\pm I\}$ or there is an index 2 subgroup π_0 of $\pi_1(M)$ such that $\rho|\pi_0$ is reducible for each $\rho \in \bar{t}^{-1}(X_0)$. Notice that even if the first case arises, we can still find such a $\pi_0 \subset \pi_1(M)$. This is because P would factor through $\pi_1(M)/\pi_1(S) \cong \mathbf{Z}$ or $\mathbf{Z}/2 * \mathbf{Z}/2$ and so there is an index 2 subgroup $\pi_0 \subset \pi_1(M)$ for which $P|\pi_0$ is reducible. That $\rho|\pi_0$ is reducible for each $\rho \in \bar{t}^{-1}(X_0)$ now follows from Lemma 4.6 (3) of [BZ2]. In either case we see that ρ_0 restricts to a reducible representation on some subgroup of index 2 in $\pi_1(M)$, which is clearly impossible as $\mathcal{B}$ is assumed to be a 2-dimensional hyperbolic orbifold. Thus S must compress in $M(r_0)$. Then by choice of S, it is incompressible in $M(r)$ for each slope r for which $\Delta(r, r_0) > 1$. Furthermore observe that as r_0 is a toral boundary slope, the work of Y.-Q. Wu [Wu2] and S. Oh [Oh] implies that $M(r)$ is irreducible as long as $\Delta(r, r_0) \geq 4$. Thus $M(r)$ is Haken when $\Delta(r, r_0) \geq 4$, and so Theorem 0.7 holds in this final case. □

§3. Examples

We present several families of examples of virtual Haken Dehn surgery in this section.

Example 3.1. Let $K(p, q, 2r)$ be a pretzel knot in S^3 (thus both p and q must be odd) with $\gcd(p, q) = d > 1$ and let M be the exterior of K in S^3. By [Oe, Corollary 4], M is small and so by the main result of [Ha], all Dehn fillings on M, except possibly for finitely many, yield non-Haken manifolds. The filling on M with slope $2(p + q)$ yields a manifold whose fundamental group surjects onto $\pi_1^{orb}(D^2(2, d)) \cong Z_2 * Z_d$ [S, p. 41-46]. By Theorem 0.5, all slopes m/n with $|m - 2(p + q)n| > 6$ yield manifolds whose fundamental groups are virtually $\mathbf{Z}*\mathbf{Z}$-representable. In fact if $d \geq 7$, the addendum to Theorem 0.5 implies that this holds for all slopes m/n with $|m - 2(p + q)n| > 2$. If K is not fibred, then combining these observations with Theorem 0.4 implies that there are at most finitely many non virtually Haken filling slopes on ∂M.

To construct a more specific set of examples, let $p \geq 7$ be an odd integer and set $K = K(p, p, 2r)$ where $|r| \geq 2$. Then K is not fibred [Ga4, §6] and so must be hyperbolic (recall from above that K is simple). Now pretzel knots satisfy the cabling conjecture [LZ, Proposition], and so all fillings on M yield irreducible manifolds. Consider a slope r of K corresponding to the rational fraction m/n where without loss of generality $n \geq 0$. Since

$d = p \geq 7$, we therefore have that $M(m/n)$ is virtually Haken as long as

$$|m - 4pn| > 2. \tag{5}$$

The genus of K, say g, is bounded above by the genus of the spanning surface given by the Seifert algorithm, so $g \leq p$. On the other hand, g is bounded below by 1/2 the degree of the Alexander polynomial of K, which also evaluates to p (see [Ma, Proposition 14]). Thus $g = p$ and so Theorem 0.4 shows that $K(m/n)$ is virtually Haken as long as

$$|m| < 12p - 4. \tag{6}$$

Inequalities (5) and (6) give $4pn - 2 \leq m \leq 4pn + 2$, and so we have $1 \leq n \leq 2$. Now $M(4p)$ is itself a Haken manifold, so it follows that if $M(r)$ is not virtually Haken, then r is contained among the seven slopes corresponding to the fractions

$$\{1/0, (4p-2), (4p-1), (4p+1), (4p+2), (8p-1)/2, (8p+1)/2\}.$$

Example 3.2. Another interesting set of examples occurs for the pretzel knots $K = K(p, -p, 2r)$ where $p \geq 7$ is an odd integer and $|r| \geq 2$. These knots are fibred [Ga4, §6], so we cannot apply Theorem 0.4, the result of Cooper and Long. Nevertheless Theorem 0.5 does apply with respect to the longitudinal slope $0 = 2(p + (-p))$. As $d = p \geq 7$, $M(m/n)$ are virtually $\mathbf{Z} * \mathbf{Z}$-representable as long as $|m| > 2$. We also observe that M must be hyperbolic, for longitudinal surgery on a torus knot does not admit a surjection onto the group $\mathbf{Z}/2 * \mathbf{Z}/p$. Therefore $M(m/n)$ is virtually Haken whenever $|m| > 2$.

Example 3.3. Consider Dehn surgery on the Borromean link in S^3. Let N denote the exterior of the link and let $N(m_1/n_1, m_2/n_2, m_3/n_3)$ denote the manifold obtained by Dehn filling the three toral boundary components T_i of N with slope m_i/n_i, i=1,2,3. (Here the slopes are parameterized by the standard meridian-longitude coordinates.) Then for all triples $(m_1/n_1; m_2/n_2; m_3/n_3)$ with $|m_1|, |m_2| > 4$, $|n_1|, |n_2| > 5$ and $|n_3| > 1$, $N(m_1/n_1, m_2/n_2, m_3/n_3)$ is virtually Haken and contains free subgroup of rank two. Note that most of these manifolds are hyperbolic and non-Haken.

The justification goes as follows. By [Th2], the complement of the link is hyperbolic. By [Go, Theorem 1.3], $N(m_1/n_1; \emptyset; \emptyset)$ (where the symbol $\emptyset$ means leaving the boundary torus open with doing filling) is hyperbolic if $|n_1| > 5$ since $N(1/0; \emptyset; \emptyset)$ is not hyperbolic. Likewise, $N(m_1/n_1; m_2/n_2; \emptyset)$ is hyperbolic if $|n_2| > 5$ since $N(m_1/n_1; 1/0; \emptyset)$ is not hyperbolic. Since $N(m_1/n_1; m_2/n_2; 1/0)$ is a connected sum of two lens spaces of orders $|m_1|$ and $|m_2|$, we may apply Theorem 0.6 to deduce that $N(m_1/n_1, m_2/n_2, m_3/n_3)$ is virtually Haken and contains a free subgroup of rank 2 for all triples $(m_1/n_1; m_2/n_2; m_3/n_3)$ with $|m_1|, |m_2| > 2$, $|n_1|, |n_2| > 5$ and $|n_3| > 1$. If we further assume that $|n_3| > 22$, then the fillings yield hyperbolic manifolds [BH]. Note that the exterior of the Borromean link contains no closed incompressible non-boundary parallel surface. Hence most of the manifolds $N(m_1/n_1, m_2/n_2, m_3/n_3)$ are non-Haken by a result of Hatcher [Ha]. One can construct many more such examples by considering Dehn surgery on Brunnian links.

References

[Ba] M. Baker, *On coverings of figure eight knot surgeries*, Pac. J. Math. **150** (1991), 215-228.

[BMS] G. Baumslag, J. Morgan and P. Shalen, *Generalized triangle group*, Math. Proc. Camb. Phil. Soc. **102** (1987), 25-31.

[BP] B. Baumslag and S. Pride, *Groups with two more generators than relators*, J. Lond. Math. Soc. (2) **17** (1978), 425-426.

[Be] A. F. Beardon, *The geometry of discrete groups*, GTM **91**, Springer-Verlag, 1983.

[BH] S. Bleiler and C. Hodgson, *Spaces forms and Dehn fillings*, Topology **35** (1996), 809-833.

[BCSZ] S. Boyer, M. Culler, P. Shalen and X. Zhang, in preparation.

[BZ1] S. Boyer and X. Zhang, *Finite Dehn surgery on knots*, J. Amer. Math. Soc. **9** (1996), 1005-1050.

[BZ2] —, *On Culler-Shalen seminorms and Dehn filling*, to appear in the Ann. of Math.

[CJ] A. Casson and D. Jungreis, *Convergence groups and Seifert fibred* 3*-manifolds*, Invent. Math. **118** (1994), 441-456.

[CL] D. Cooper and D. Long, *Virtually Haken Dehn-filling*, preprint.

[CGLS] M. Culler, C. Gordon, J. Luecke and P. Shalen, *Dehn surgery on knots*, Ann. of Math. **125** (1987), 237-300.

[Ga1] D. Gabai, *Homotopy hyperbolic 3-manifolds are virtually hyperbolic*, J. Amer. Math. Soc. **7** (1994), 193-198.

[Ga2] —, *Convergence groups are Fuchsian groups*, Ann. Math. **136** (1992), 447-510.

[Ga3] —, *Foliations and the topology of* 3*-manifolds II*, J. Diff. Geo. **26** (1987), 461-478.

[Ga4] —, *Detecting fibred links in* S^3, Comm. Math. Helv. **61** (1986), 519-555.

[GMT] D. Gabai, R. Meyerhoff and N. Thurston, *Homotopy hyperbolic 3-manifolds are hyperbolic*, preprint.

[Go] C. Gordon, *Boundary slopes of punctured tori in* 3*-manifolds*, to appear in the Trans. Amer. Math. Soc.

[GL] C. Gordon and J. Luecke, *Reducible manifolds and Dehn surgery*, Topology **35** (1996), 385-409.

[Ha] A. Hatcher, *On the boundary curves of incompressible surfaces*, Pacific J. Math. **99** (1982), 373-377.

[He1] J. Hempel, *3-Manifolds*, Annals of Math. Studies. **86**, Princeton University Press, 1976.

[He2] J. Hempel, *Coverings of Dehn fillings of surface bundles*, Top. Appl. **24** (1986), 157-170.

[Ja] W. Jaco, *Lectures on three-manifolds topology*, CBMS Regional Conf. Ser. Math. **43**, Amer. Math. Soc., 1980.

[KL] S. Kojima and D. Long, *Virtual Betti numbers of some hyperbolic* 3*-manifolds*, in Y. Matsumoto, T. Mizutani, & S. Morita (Eds.), *A Fete of Topology*, Academic Press, 1988.

[LZ] E. Luft and X. Zhang, *Symmetric knots and the cabling conjecture*, Math. Ann. **298** (1994), 489-496.

[Ma] N. Maruyama, *On Dehn surgery along a certain family of knots*, J. Tsuda College **19** (1987), 261-280.

[Mo] L. Moser, *Elementary surgery along a torus knot*, Pac. J. Math. 38 (1971) 737-745.

[N] A. Nicas, *An infinite family of hyperbolic non-Haken* 3*-manifolds with vanishing Whitehead groups*, Math. Proc. Camb. Phil. Soc. **99** (1986), 239-246.

[Oe] U. Oertel, *Closed incompressible surfaces in complements of star links*, Pac. J. Math. **111** (1984), 209-230.

[Oh] S. Oh, *Reducible and toroidal 3-manifolds obtained by Dehn filling*, Top. Appl., to appear.

[R] D. Rolfsen, *Knots and Links*, 2nd edition, Publish or Perish, 1990.

[S] J. Simon, *Methods for proving that certain classes of knots have property P*, Ph. D. thesis, University of Wisconsin, 1969.

[Th1] W. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357-381.

[Th2] —, *The Geometry and Topology of Three Manifolds*, Princeton University Lecture Notes, 1979.

[Wa] F. Waldhausen, *The word problem in fundamental groups of sufficiently large* 3*-manifolds*, Ann. Math. **88** (1968), 272-280.

[Wu1] Y.-Q. Wu, *Incompressibility of surfaces in surgered* 3*-manifolds*, Topology **31** (1992), 271-279.

[Wu2] —, *Dehn surgeries producing reducible manifold and toroidal manifold*, preprint.

[Wu3] —, *Dehn surgery and simple manifolds*, preprint.

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